On matrix generalization of Hurwitz polynomials

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Contents

1 Introduction

This thesis focuses on matrix generalizations of the classical notion of Hurwitz polynomials. A polynomial with all its roots in the open left half plane of the complex plane is called a Hurwitz polynomial in honour of the German mathematician A. Hurwitz. The study of Hurwitz polynomials is an essential branch of the research of zero localization problem. It has a long and abundant history (see e.g. the monograph by Gantmacher [37, Chapter XV, pp. 172–250]). These polynomials were studied in 1856 by Hermite [43] using the theory of the Cauchy indices. Nevertheless, the numerous applications to the stability of dynamical systems [65] required an algorithm for verification of the Hurwitz property based on the coefficients of polynomials. This criterion was obtained by Routh [68] in 1877. Independently of Routh's work, later in 1895 Hurwitz [47] provided some determinantal inequalities as a criterion for such polynomials, which is called Routh-Hurwitz criterion nowadays. In 1914, Liénard and Chipart [61] developed a more efficient criterion than the Routh-Hurwitz criterion with less determinantal inequalities. In 1945, Wall [70, Theorem A] presented an equivalent condition for a real polynomial to be a Hurwitz polynomial in terms of the coefficients of the related Jacobi continued fraction. Gantmacher [37, Chapter XV] gave a comprehensive overview of issues related to Hurwitz polynomials: their connections to the Lyapunov stability, the Sturm sequences, the Cauchy indices, the Stieltjes continued fractions and moment problems, the Hankel and Hurwitz matrices, quadratic forms.

The importance of matrix polynomials is quite clear from a joint monograph by Gohberg, Lancaster, Rodman [38] in 1982, which provided a comprehensive treatment of the theory of matrix polynomials. The zero location problems returned to the spotlight owing to the intrinsic development of matrix polynomial theory. Consider a $p \times p$ matrix polynomial F of degree n

$$F(z) = \sum_{k=0}^{n} A_k z^{n-k}, \quad z \in \mathbb{C}.$$

where $A_k \in \mathbb{C}^{p\times p}$, $k \in \mathbb{Z}_{0,n}$. We call a zero of the determinant polynomial $\det F$ a zero of matrix polynomial F. In this regard, it is natural to define the direct matricial generalization of Hurwitz polynomials as follows: A $p \times p$ matrix polynomial F is called a Hurwitz matrix polynomial if $\det F$ is a Hurwitz polynomial.

In comparison to the scalar case, to study criteria for Hurwitz matrix polynomials F which do not require the computation of $\det F$ appears much more complicated. Some obstacles occur when one tries to adopt the approaches used for the study of zero location in the scalar case. On one hand, the theory of Cauchy indices seems to lose its effectiveness on account of no suitable matrix version of the fundamental theorem of algebra existing, which yields correlations between the matrix coefficients of a matrix polynomial and its zeros. On the other hand, there are some limitations of the determinant approach to some algebraic constructs involving the matrix coefficients of a matrix polynomial such that no satisfactory matricial extension of the Routh-Hurwitz criterion or Liénard-Chipart test exists.

Recently an alternative method to explore Hurwitz matrix polynomials was developed by Choque Rivero [17], who followed another line of matricial extensions of the classical Hurwitz polynomial. To explain this generalized Hurwitz polynomial, we start with a classical result which characterized scalar Hurwitz polynomials via the Stieltjes continued fraction (see e.g. Gantmacher [37, Theorem 16, p232]):

Given a real polynomial F of degree n, F is a Hurwitz polynomial if and only if the even part $F_{\langle e\rangle}$ and the odd part $F_{\langle o\rangle}$ of F admit the following Stieltjes continued fraction:

$$\frac{F_{\langle o\rangle}(z)}{F_{\langle e\rangle}(z)} = d_{-1} + \cfrac{1}{zc_0 + \cfrac{1}{\ddots\, d_{m-2} + \cfrac{1}{zc_{m-1} + d_{m-1}^{-1}}}}, \tag{1.1}$$

where $d_{-1} \geq 0$, $d_k > 0$ and $c_k > 0$ for each $k \in \mathbb{Z}_{0,m-1}$.

The generalized matricial Hurwitz polynomial introduced in [17], called right matrix Hurwitz type polynomial, is defined for a matrix polynomial F for which the even part $F_{\langle e\rangle}$ and the odd part $F_{\langle o\rangle}$ follow the matrix analogue of a Stieltjes continued fraction as in (1.1) (see Definition 8.1). One of most essential features of a matrix Hurwitz type polynomial is the inheritance of an elegant bijective correspondence between real Hurwitz polynomials and Stieltjes positive definite sequences (see Gantmacher [37, Theorem 18]), which is shown in Theorem 7.10 of [17]:

Given a monic $p \times p$ matrix polynomial F of degree n, F is a matrix Hurwitz type polynomial if and only if the $(n-1)$-th section $\mathscr{S}_{\langle n-1\rangle}$ (see (3.2)) of right Markov parameters of F (see Definition 7.2) is a Stieltjes positive definite sequence.

However, we must note that the notion "matrix Hurwitz type polynomial" is still irrelative to "Hurwitz matrix polynomial" due to the totally unclear zero location of the former notion. So there is an open question that remains to be considered: What is the connection between these two notions "matrix Hurwitz type polynomial" and "Hurwitz matrix polynomial"?

In view of the unsatisfactory state of the art, the main goal of this thesis is to provide some criteria to identify Hurwitz matrix polynomials and discover its relation to the "matrix Hurwitz-type polynomials".

The author's motivation originated from the recent developments of the research on Stieltjes positive definite matrix sequences and finite matrix Stieltjes moment problem by the Leipzig Schur analysis group around Fritzsche and Kirstein. Fritzsche, Kirstein and Mädler [34], [35] set up a Schur-type algorithm for matrix sequences to obtain a complete description of the solution set of the truncated matricial Stieltjes moment problem in the most general case. This Schur-type algorithm for Stieltjes positive definite matrix sequences is adopted by Choque Rivero and Mädler [19] to give an explicit representation of orthogonal matrix polynomials.

Given a $p \times p$ matrix polynomial F, we use the so-called Markov parameters approach to explore some related matrix sequences. More specifically, we start with the left and right types of matrix rational functions formed by the even part $F_{\langle e\rangle}$ and the odd part $F_{\langle o\rangle}$. It should be mentioned that both dual types generally call for our consideration owing to the non-commutativity of matrix multiplication, although their properties are mostly symmetric. Then the extended sequence of right (resp. left) Markov parameters (for short SRMP (resp. SLMP)) of a matrix polynomial are formed by the matrix coefficients of the Laurent series of the formerly constructed dual types of matrix rational functions. We furthermore investigate the features of matricial Markov parameters to give equivalent conditions for a matrix polynomial being a Hurwitz matrix polynomial.

To get access to the inner structure of the matricial Markov parameters of a matrix polynomial, we decide to bring forth the tool of matrix fraction description for transfer function matrices. It transpires that matrix fraction description plays an important role in the "polynomial systems theory" initiated by some pioneers such as Rosenbrock [69], Popov [67] and Wolovich [71] et al. This approach avoids the lack of transparency in the state-space method and, moreover, obtains a natural generalization from the single input-single output transfer function to multiple input-multiple output cases.

Note that, up to the author's knowledge, there are no ample results connecting matrix sequences of Markov parameters and the zero localization of matrix polynomials. Here, a key auxiliary tool we use to bridge these two subjects is a matricial version of the Hermitian-Fujiwara theorem by Lerer/Tismenetsky [60], which is based on the spectral theory of matrix polynomials (see a comprehensive study in the monograph [38]). The formula in this result is given in terms of an Anderson-Jury Bezoutian matrix related to the given matrix polynomial, which is a matricial generalization of the standard Bezoutian matrix. The composition for this instrument is a little strict in the sense that it requires a specific relation to be satisfied for a triple of matrix polynomials. Our particular interest here is a special case of this Bezoutian matrix, which is connected with Hermitian transfer function matrices.

By adopting the approaches highlighted above, we are able to derive some important consequences concerning Hurwitz matrix polynomials and corresponding topics:

- As a preliminary outcome, given a para-Hermitian strictly proper transfer function matrix G we establish interesting relations between the Markov parameter sequence of G and Hurwitz matrix polynomials linked with the matrix fraction description of G.

- We give solutions to the matricial Routh-Hurwiz problem in terms of the truncated SLMP or SRMP of matrix polynomials.

- We achieve our main goal to demonstrate that F is a Hurwitz matrix polynomial if and only if the truncated SLMP or SRMP of F is a Stieltjes positive definite sequence. Moreover we discover that the two notions "matrix Hurwitz type polynomial" and "Hurwitz matrix polynomial" are equivalent.

- We represent a Hurwitz matrix polynomial F via a three-terms recurrence relation, where the matrix coefficients are given from the Hurwitz parametrization of F.
- We build a correspondence between Hurwitz matrix polynomials and the Stieltjes quadruple of sequences of left orthogonal matrix polynomials.
- We look for the correspondences between the existence of quasi-stable matrix polynomials, the solvability of truncated Stieltjes moment problems and multiple Nevanlinna-Pick interpolation in the Stieltjes class.
- We give a sufficient and necessary condition for a matrix polynomial to be a quasi-stable matrix polynomial.
- We find that the SLMP and SRMP of a quasi-stable matrix polynomial or a Hurwitz matrix polynomial of degree n are completely degenerate sequences of order n.

Now we conclude this introduction with the outline of the thesis.

Chapters 2–4 serve as a delicate framework of the whole thesis, providing some basic algebraic knowledge and techniques from the theory of matrix polynomials, univariate moment theory and the "polynomial systems theory" in multivariable linear systems.

In Chapter 2, we start with preliminaries from the theory of matrix polynomials. Much attention is paid to greatest common divisors of two matrix polynomials. Some properties of these are shown in connection to the spectrum and formulation of the greatest common divisors and coprimeness.

Chapter 3 turns to a short explanation of several types of matrix sequences. Some algebraic constructs, generated by these matrix sequences, are also included. We emphasize here the linear-algebraic properties of these objects which give rise to the criteria for solvability of matricial Hamburger moment problems and matricial Stieltjes moment problems.

Chapter 4 is divided into two sections. Section 4.1 starts by introducing several types of transfer function matrices and their matrix fraction descriptions. We focus on two dual types of matrix fraction descriptions, which are called normalized left matrix fraction description (for short NLFD) and normalized right matrix fraction description (for short NRFD), for the study of strictly proper transfer function matrices. The first advantage of the NLFD and NRFD is that there exists a bijective map between the NLFD (resp. NRFD) and a strictly proper transfer function matrix (see Proposition 4.9). Some more important features appear in the realization of NLFD and NRFD from the sequence of Markov parameters of strictly proper transfer function matrices (see Propositions 4.13 and 4.14) and in the characterization of Hermitian strictly proper transfer function matrices via the extended sequence of

Markov parameters (for short SMP) (see Proposition 4.15). Section 4.2 should be viewed as a continuation of Section 4.1, where the theory of monic orthogonal right (resp. left) system of matrix polynomials is added. Given a particular sequence of Hermitian transfer function matrices $\{G_k\}_{k=1}^{m}$, we further seek the correspondence between the NLFD (resp. NRFD) of G_k and a monic orthogonal system of matrix polynomials with respect to the truncated Markov parameters of G_k (see Proposition 4.28).

The main part of Chapter 5 is to revisit a matricial refinement of the classical Hermite-Fujiwara theorem by Lerer/Tismenetsky [60] (see Theorem 5.17). This theorem is performed to count the inertia of a matrix polynomial with respect to $\mathbb{R}$, i.e., how many zeros of a given matrix polynomial lie on the open upper half plane and the open lower half plane, whose common boundary is $\mathbb{R}$. The formula in this result is given in terms of a Anderson-Jury Bezoutian matrix related to the given matrix polynomial. Our particular interest here is a special case of this Bezoutian matrix, which is connected with a Hermitian transfer function matrix. It turns out in Proposition 5.12 that this special Anderson-Jury Bezoutian matrix is congruent to a block Hankel matrix generated by the SMP of the Hermitian transfer function matrix.

In Chapter 6 we seek a preliminary outcome for Hurwitz matrix polynomial. Propositions 6.3 and 6.4 check whether a matrix polynomial is a Hurwitz matrix polynomial or not by its SMP when this matrix polynomial is confined to be a linear combination of a NRFD for a para-Hermitian strictly proper transfer function matrix.

The matricial Routh-Hurwitz problem is the main concern in Chapter 7, which determines the inertia of matrix polynomials with respect to $\mathrm{i}\mathbb{R}$, that is to find out how many zeros of a given matrix polynomial lie on the open left half plane and the open right half plane, respectively. Theorem 7.16 and its refinement Theorem 7.17 give solutions to the matricial Routh-Hurwitz problem in terms of the truncated SRMP of matrix polynomials. The dual version of solutions in terms of the truncated SLMP of matrix polynomials are given in Theorems 7.18 and 7.19.

As is mentioned above, when compared to Hurwitz matrix polynomial, the notion “matrix Hurwitz type polynomial” is another line of matricial extensions of Hurwitz polynomial. The main objective in Chapter 8 is to seek a three-term recurrence relation for this type of matrix polynomial for a given Hurwitz parametrization (see Lemma 8.3). It becomes transparent that the matrix polynomial constructed in this three-term procedure is the unique matrix Hurwitz type polynomial for a given Hurwitz parametrization (see Theorem 8.11).

With these preparations, our main results are obtained in Chapter 9. Based on Chapter 6, we firstly describe what conditions must be imposed on the SRMP or SLMP in order to ensure that the corresponding matrix polynomial is a Hurwitz matrix polynomial (see Theorems 9.1 and 9.2). As a consequence, we reveal in Theorems 9.7 and 9.8 that the two notions “matrix Hurwitz type polynomials” and “Hurwitz matrix polynomials” are equivalent. With the fulfilment of the recurrence relation for matrix Hurwitz type polynomials, we develop a three-term recurrence formula for left (resp. right) $\mathscr{S}$-system of Hurwitz matrix polynomials (see Theorem 9.13 (resp. Theorem 9.15)). This left (resp. right) $\mathscr{S}$-system of Hurwitz matrix polynomials $(F_k)_{k=1}^m$ is characterized by the fact that for each $k \in \mathbb{Z}_{1,m-1}$, the $(k+1)$-th SLMP (resp. SRMP) of F_{k+1} is a one-step extension of the k-th SLMP (resp. SRMP) of F_k. This class of Hurwitz matrix polynomials also involves the interesting property that the matrix rational function formed by F_k and F_{k+1} admits a certain finite matrix continued fraction of Jacobi type (see Proposition 9.19) and the important Christoffel-Darboux relations hold for $(F_k)_{k=1}^m$ (see Propositions 9.20 and 9.21). We obtain a sufficient and necessary condition for a $\mathscr{S}$-system of Hurwitz matrix polynomials being one-step extendable (see Theorems 9.24 and 9.25). Moreover, a bijective correspondence between the unique left $\mathscr{S}$-system of Hurwitz matrix polynomials and the Stieltjes quadruple of sequences of left orthogonal matrix polynomials is established via sequences of Markov parameters (see Theorem 9.27). Accordingly, we can give a new three-term recurrence for this Stieltjes quadruple (see Theorem 9.28).

In Chapter 10 we study another important set of matrix polynomials appearing in the theory of stability, called quasi-stable matrix polynomials. We note that these polynomials also contain Hurwitz matrix polynomials as a special case. Theorem 10.4 constructs a special quasi-stable matrix polynomial in terms of a monic orthogonal left system of matrix polynomials with respect to a Stieltjes nonnegative definite extendable sequence. Consequently, we look for the correspondences between quasi-stable matrix polynomials, Stieltjes moment problems and multiple Nevanlinna-Pick interpolation in the Stieltjes class (see Theorems 10.6 and 10.8).

It should be mentioned that the requirement that the $(n-1)$-th SRMP or SLMP of F is a Stieltjes nonnegative definite extendable sequence is not sufficient enough for F to be a quasi-stable matrix polynomial. Taking this fact into consideration, we prove that F is a quasi-stable matrix polynomial if and only if $(n-1)$-th SRMP (resp. SLMP) of F is a Stieltjes non-negative definite extendable sequence and the zeros of a greatest right (resp. left) common divisor of $F_{\langle e\rangle}$ and $F_{\langle o\rangle}$ are located on the interval $(-\infty, 0]$. In this case, it is also indicated that the SRMP (resp. SLMP) of F is completely degenerate of order n. In this regard, the whole picture of the SRMPs (resp. SLMPs) of quasi-stable matrix polynomials and Hurwitz matrix polynomials is completely described.

2 Matrix polynomials and greatest common divisors

In Section 2.1 we provide some preliminary facts relating to matrix polynomials. For a more comprehensive treatment of the theory of matrix polynomials and the original treatises, we refer the reader to [38, 40, 49] and the references therein.

Section 2.2 focuses on the study of greatest common divisors of matrix polynomials. The study of common divisors of matrix polynomials arises naturally in connection with several important matricial extensions of linear constructs for scalar polynomials such as Resultants and Bezoutians (see [4, 38, 39]) and multi-input multi-output linear systems (see [49]). We consider several parallel questions to greatest common divisors in the scalar case, such as the relation between greatest common divisors, the expression and formation of greatest common divisors from two matrix polynomials and consequences of greatest common divisors when several types of transformations are applied to two matrix polynomials.

2.1 Preliminaries

Let us first introduce some notation. Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}_0$ and $\mathbb{N}$ denote the set of all complex numbers, the set of all real numbers, the set of all nonnegative integers, and the set of all positive integers, respectively. Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. For $j \in \mathbb{N}_0$ and $l \in \mathbb{N}_0 \cup \infty$, let

$$\mathbb{Z}_{j,l} := \begin{cases} \{k \in \mathbb{N}_0 : j \leq k \leq l\}, & j \leq l \\ \emptyset, & j > l \end{cases}.$$

Let p, $q \in \mathbb{N}$. $\mathbb{C}^{p\times q}$, $\mathbb{C}_H^{p\times p}$ and $\mathbb{C}_>^{p\times p}$ stand for the set of all complex $p \times q$ matrices, the set of all complex Hermitian $p \times p$ matrices and the set of all positive definite Hermitian $p\times p$ matrices, respectively. Let $0_{p\times q}$ be the zero matrix in $\mathbb{C}^{p\times q}$ and let I_p be the identity matrix in $\mathbb{C}^{p\times p}$. For simplicity we write 0_p for $0_{p\times p}$. Let $A \in \mathbb{C}^{p\times p}$. We denote the adjoint matrix of A by $\mathrm{Adj}(A)$. The Moore-Penrose inverse of A, denoted by $A^\dagger$, is the unique solution X of the matrix equations: $AXA = A$, $XAX = X$, $AX = (AX)^*$ and $XA = (XA)^*$.

For p, $q \in \mathbb{N}$, $\mathcal{P}_{p\times q,\mathbb{C}}$ will denote the set of all $\mathbb{C}^{p\times q}$-valued polynomials F: $\mathbb{C} \to \mathbb{C}^{p\times q}$. For each $F \in \mathcal{P}_{p\times q,\mathbb{C}}$ and $k \in \mathbb{N}$, $\deg F$, the degree of F, is defined as the maximum of the degrees of its entries, and $\deg_k F$, the k-th column degree of F, as the maximum of the degrees of its entries in the k-th column. For $n \in \mathbb{N}_0$, $\mathcal{P}_{p\times q,n,\mathbb{C}}$ is the set of all $\mathbb{C}^{p\times q}$-valued polynomials of degree n. Each $F \in \mathcal{P}_{p\times q,n,\mathbb{C}}$ can be written as follows:

$$F(z) = \sum_{k=0}^{n} A_k z^{n-k}, \quad z \in \mathbb{C}. \tag{2.1}$$

where $A_k \in \mathbb{C}^{p\times q}$, $k \in \mathbb{Z}_{0,n}$. Particularly in the case $p = q$, we call F monic if $A_0 = I_p$ and call F comonic if $A_n = I_p$. Let $F \in \mathcal{P}_{p\times q,n,\mathbb{C}}$ be as in (2.1). Then we define $F^\vee : \mathbb{C} \to \mathbb{C}^{q\times p}$ as

$$F^\vee(z) := \sum_{k=0}^{n} A_k^* z^{n-k}, \quad z \in \mathbb{C}.$$

Obviously, $F^\vee \in \mathcal{P}_{q\times p,n,\mathbb{C}}$.

Definition 2.1. Suppose that $F \in \mathcal{P}_{p\times p,\mathbb{C}}$ and $\Delta^F(z) = \mathrm{diag}(z^{-\deg_1 F}, \ldots, z^{-\deg_n F})$. Then $\Gamma(F) := \lim_{z\to\infty} F(z)\Delta^F(z)$ is called the *leading column coefficient matrix of* F. F is called *column reduced* if $\Gamma(F)$ is nonsingular.

Remark 2.2. Let $F \in \mathcal{P}_{p\times p,\mathbb{C}}$ be a regular matrix polynomial as in (2.1). Then F is column-reduced if and only if A_0 is nonsingular.

Definition 2.3. For $\lambda \in \mathbb{C}$, we say that λ is a *zero of* F if $\det F(\lambda) = 0$, and the *multiplicity of* λ as a zero of F is the multiplicity of λ as a zero of $\det F(z)$.

Let $\sigma(F)$ stand for the spectrum of F, or equivalently the set of all zeros of F, determined by

$$\sigma(F) := \{z \in \mathbb{C} : \det F(z) = 0\}.$$

We say that F is *regular* (or *nonsingular*) if $\sigma(F)$ is a finite subset of $\mathbb{C}$. Otherwise F is called *singular* if $\det F(z) \equiv 0$.

Lemma 2.5 of [17, Section 2] reveals that if F is monic, then F is regular.

Definition 2.4. Let $F \in \mathcal{P}_{p\times p,\mathbb{C}}$. F is called *unimodular* if $\det F(z)$ never vanishes in $\mathbb{C}$.

Remark 2.5. Let F, $\widetilde{F} \in \mathcal{P}_{p\times p,\mathbb{C}}$ be unimodular and let $\alpha \in \mathbb{C}\backslash\{0\}$. Then αF and $F \cdot \widetilde{F}$ are unimodular.

Remark 2.6. (see [49, Lemma 6.3-1]) Let $F \in \mathcal{P}_{p\times p,\mathbb{C}}$. F is unimodular if and only if $F^{-1} \in \mathcal{P}_{p\times p,\mathbb{C}}$.

2.2 Greatest common divisors of matrix polynomials

This section is mainly relevant to dual types of greatest common divisors of two matrix polynomials which arise from the non-commutativity of matrix multiplication. We begin with an introduction of these dual types of greatest common divisors.

Definition 2.7. Let $p, q, r \in \mathbb{N}$. Let $F \in \mathcal{P}_{p\times q,\mathbb{C}}$ and let $L \in \mathcal{P}_{r\times q,\mathbb{C}}$ (resp. $\mathcal{P}_{p\times r,\mathbb{C}}$). Then F is called a *left* (resp. *right*) *multiple* of L if there exists a matrix polynomial $M \in \mathcal{P}_{p\times r,\mathbb{C}}$ (resp. $\mathcal{P}_{r\times q,\mathbb{C}}$) such that, for $z \in \mathbb{C}$,

$$F(z) = M(z)L(z) \quad (\text{resp. } F(z) = L(z)M(z)).$$

In this case, L is called a *right* (resp. *left*) *divisor* of F.

Definition 2.8. Let $p, q \in \mathbb{N}$. Let $F \in \mathcal{P}_{q\times p,\mathbb{C}}$ (resp. $\mathcal{P}_{p\times q,\mathbb{C}}$). Let $L \in \mathcal{P}_{p\times p,\mathbb{C}}$ and let $\widetilde{L} \in \mathcal{P}_{q\times p,\mathbb{C}}$ (resp. $\mathcal{P}_{p\times q,\mathbb{C}}$). We call F a *left* (resp. *right*) *common multiple of* L *and* $\widetilde{L}$ if there exists a pair of matrices $M \in \mathcal{P}_{q\times p,\mathbb{C}}$ (resp. $\mathcal{P}_{p\times q,\mathbb{C}}$) and $\widetilde{M} \in \mathcal{P}_{q\times q,\mathbb{C}}$ such that, for $z \in \mathbb{C}$,

$$\begin{aligned} &F(z) = \widetilde{M}(z)\widetilde{L}(z) = M(z)L(z) \\ &(\text{resp. } F(z) = \widetilde{L}(z)\widetilde{M}(z) = L(z)M(z)). \end{aligned}$$

Definition 2.9. Let $p, q \in \mathbb{N}$. Let $F \in \mathcal{P}_{p\times q,\mathbb{C}}$ (resp. $\mathcal{P}_{q\times p,\mathbb{C}}$) and let $\widetilde{F} \in \mathcal{P}_{p\times p,\mathbb{C}}$. Let also $L \in \mathcal{P}_{p\times p,\mathbb{C}}$. Then L is called a *left* (resp. *right*) *common divisor of* F *and* $\widetilde{F}$ if there exists a pair of matrix polynomials $M \in \mathcal{P}_{p\times q,\mathbb{C}}$ (resp. $\mathcal{P}_{q\times p,\mathbb{C}}$) and $\widetilde{M} \in \mathcal{P}_{p\times p,\mathbb{C}}$ such that, for $z \in \mathbb{C}$,

$$F(z) = L(z)M(z), \quad \widetilde{F}(z) = L(z)\widetilde{M}(z) \tag{2.2}$$

$$(\text{resp. } F(z) = M(z)L(z), \quad \widetilde{F}(z) = \widetilde{M}(z)L(z)). \tag{2.3}$$

L is called a *greatest left* (resp. *right*) *common divisor*, or for short *g.l.c.d* (resp. *g.r.c.d*) *of* F *and* $\widetilde{F}$ if any other left (resp. right) common divisor is a left (resp. right) divisor of L.

Definition 2.10. Suppose that F and $\widetilde{F} \in \mathcal{P}_{p\times p,\mathbb{C}}$. The matrix polynomials F and $\widetilde{F}$ are called *right* (resp. *left*) *coprime* if all g.r.c.ds (resp. g.l.c.ds) of F and $\widetilde{F}$ are unimodular.

Remark 2.11. Let $p, q \in \mathbb{N}$. Let $F \in \mathcal{P}_{p\times q,\mathbb{C}}$ (resp. $\mathcal{P}_{q\times p,\mathbb{C}}$) and let $\widetilde{F} \in \mathcal{P}_{p\times p,\mathbb{C}}$. Let $L \in \mathcal{P}_{p\times p,\mathbb{C}}$ be a left (resp. right) common divisor of F and $\widetilde{F}$ such that the equality (2.2) (resp. (2.3)) holds for a certain pair of matrix polynomials $M \in \mathcal{P}_{p\times q,\mathbb{C}}$ (resp. $\mathcal{P}_{q\times p,\mathbb{C}}$) and $\widetilde{M} \in \mathcal{P}_{p\times p,\mathbb{C}}$. Then L is a g.l.c.d (resp. g.r.c.d) of F and $\widetilde{F}$ if and only if M and $\widetilde{M}$ are left (resp. right) coprime.

Remark 2.12. Let $p \in \mathbb{N}$. Let F, $\widetilde{F}$ and $L \in \mathcal{P}_{p\times p,\mathbb{C}}$.

(i) If L is a g.l.c.d of F and $\widetilde{F}$, then $\sigma(L) \subseteq \sigma(F) \cap \sigma(\widetilde{F})$.

(ii) If L is a g.r.c.d of F and $\widetilde{F}$, then $\sigma(L) \subseteq \sigma(F) \cap \sigma(\widetilde{F})$.

Remark 2.13. Let $p \in \mathbb{N}$. Let F, $\widetilde{F} \in \mathcal{P}_{p\times p,\mathbb{C}}$. If $\sigma(F) \cap \sigma(\widetilde{F}) = \emptyset$, then

(i) F and $\widetilde{F}$ is left coprime.

(ii) F and $\widetilde{F}$ is right coprime.

Proof. Suppose that $\sigma(F) \cap \sigma(\widetilde{F}) = \emptyset$ and L is a g.l.c.d (resp. g.r.c.d) of F and $\widetilde{F}$. Using Remark 2.12, we have $\sigma(L) = \emptyset$, or equivalently, L is unimodular. Hence F and $\widetilde{F}$ is left (resp. right) coprime. □

Proposition 2.14. *(see [49, pp. 377-378]) Let $p, q \in \mathbb{N}$. Let $F \in \mathcal{P}_{p\times q,\mathbb{C}}$ (resp. $\mathcal{P}_{q\times p,\mathbb{C}}$) and let $\widetilde{F} \in \mathcal{P}_{p\times p,\mathbb{C}}$. Let F_1 and F_2 be two g.l.c.ds (resp. g.r.c.ds) of F and $\widetilde{F}$.*

(i) *Suppose that F_1 is nonsingular. Then there exists a unimodular matrix polynomial $W \in \mathcal{P}_{p\times p,\mathbb{C}}$ such that for each $z \in \mathbb{C}$,*

$$\begin{aligned} &F_1(z) = F_2(z)W(z) \\ &(\textit{resp. } F_1(z) = W(z)F_2(z)). \end{aligned}$$

(ii) *If F_1 is unimodular, then F_2 is unimodular.*

(iii) *$\sigma(F_1) = \sigma(F_2)$.*

According to Proposition 2.14, we can immediately see the following connection between greatest common divisors of matrix polynomials.

Proposition 2.15. *Let $p, q \in \mathbb{N}$. Let $F \in \mathcal{P}_{p\times q,\mathbb{C}}$ (resp. $\mathcal{P}_{q\times p,\mathbb{C}}$) and let $\widetilde{F} \in \mathcal{P}_{p\times p,\mathbb{C}}$. Let F_1 is a g.l.c.d (resp. g.r.c.d) of F and $\widetilde{F}$. Let $F_2 \in \mathcal{P}_{p\times p,\mathbb{C}}$. F_2 is a g.l.c.d (resp. g.r.c.d) of F and $\widetilde{F}$ if and only if there exists a unimodular matrix polynomial $W \in \mathcal{P}_{p\times p,\mathbb{C}}$ such that for each $z \in \mathbb{C}$,*

$$\begin{aligned} &F_1(z) = F_2(z)W(z) \\ &(\textit{resp. } F_1(z) = W(z)F_2(z)). \end{aligned}$$

Proposition 2.16. *[Bezout's identity] Let $p, q \in \mathbb{N}$. Let $F \in \mathcal{P}_{p\times p,\mathbb{C}}$.*

(i) *Let $\widetilde{F} \in \mathcal{P}_{p\times q,\mathbb{C}}$. F and $\widetilde{F}$ are right coprime if and only if there exist a pair of matrix polynomials $X \in \mathcal{P}_{q\times p,\mathbb{C}}$ and $Y \in \mathcal{P}_{p\times p,\mathbb{C}}$ such that for $z \in \mathbb{C}$,*

$$\widetilde{F}(z)X(z) + F(z)Y(z) = I_p.$$

(ii) *Let $\widetilde{F} \in \mathcal{P}_{q\times p,\mathbb{C}}$. F and $\widetilde{F}$ are left coprime if and only if there exists a pair of matrix polynomials $X \in \mathcal{P}_{p\times q,\mathbb{C}}$ and $Y \in \mathcal{P}_{p\times p,\mathbb{C}}$ such that for $z \in \mathbb{C}$,*

$$X(z)\widetilde{F}(z) + Y(z)F(z) = I_p.$$

The assertion (i) of Proposition 2.16 follows from [49, Lemma 6.3-5] (see also [66]). By adopting analogue approach we can verify (ii).

Next we consider how to derive a g.r.c.d or g.l.c.d of matrix polynomials.

Proposition 2.17. *Let $p, q \in \mathbb{N}$. Let $F \in \mathcal{P}_{p\times p,\mathbb{C}}$.*

(i) *Let $\widetilde{F} \in \mathcal{P}_{q\times p,\mathbb{C}}$. Suppose that there exists a unimodular matrix polynomial $U_R \in \mathcal{P}_{(p+q)\times(p+q),\mathbb{C}}$ and some matrix polynomial $F_1 \in \mathcal{P}_{p\times p,\mathbb{C}}$ such that*

$$U_R(z) \cdot \begin{pmatrix} F(z) \\ \widetilde{F}(z) \end{pmatrix} = \begin{pmatrix} F_1(z) \\ 0_{q\times p} \end{pmatrix}. \tag{2.4}$$

Then F_1 is a g.r.c.d of F and $\widetilde{F}$.

(ii) *Conversely, let $F_1 \in \mathcal{P}_{p\times p,\mathbb{C}}$ be a g.r.c.d of F and $\widetilde{F}$. Then there exists a unimodular matrix polynomial $U_R \in \mathcal{P}_{(p+q)\times(p+q),\mathbb{C}}$ such that (2.4) holds.*

(iii) *Let $\widetilde{F} \in \mathcal{P}_{p\times q,\mathbb{C}}$. Suppose that there exists a unimodular matrix polynomial $U_L \in \mathcal{P}_{(p+q)\times(p+q),\mathbb{C}}$ and some matrix polynomial $F_1 \in \mathcal{P}_{p\times p,\mathbb{C}}$ such that*

$$\left(F(z), \widetilde{F}(z)\right) \cdot U_L(z) = \left(F_1(z), 0_{p\times q}\right). \tag{2.5}$$

Then F_1 is a g.l.c.d of F and $\widetilde{F}$.

(iv) *Conversely, let $F_1 \in \mathcal{P}_{p\times p,\mathbb{C}}$ be a g.l.c.d of F and $\widetilde{F}$. Then there exists a unimodular matrix polynomial $U_L \in \mathcal{P}_{(p+q)\times(p+q),\mathbb{C}}$ such that (2.5) holds.*

Proof. The assertion (i) is proved in [49, Lemma 6.3-3].

The proof of (ii): Suppose that $\begin{pmatrix} F(z) \\ \widetilde{F}(z) \end{pmatrix}$ is transformed into the following Hermitian form (see [49, Theorem 6.3-2]), or to say, there exists a unimodular matrix polynomial $\widetilde{U}_R \in \mathcal{P}_{(p+q)\times(p+q),\mathbb{C}}$ and $F_2 \in \mathcal{P}_{p\times p,\mathbb{C}}$ such that

$$\widetilde{U}_R(z) \cdot \begin{pmatrix} F(z) \\ \widetilde{F}(z) \end{pmatrix} = \begin{pmatrix} F_2(z) \\ 0_{q\times p} \end{pmatrix}. \tag{2.6}$$

Then due to (i) of Proposition 2.17, F_2 is a g.r.c.d of F and $\widetilde{F}$. Suppose that F_1 is a g.r.c.d of F and $\widetilde{F}$. One can see from Proposition 2.14 that there exists a unimodular $W \in \mathcal{P}_{p\times p,\mathbb{C}}$ such that

$$W(z)F_2(z) = F_1(z).$$

Substituting

$$U_R(z) := \operatorname{diag}(W(z), I_m)\widetilde{U}_R(z)$$

into (2.6), we obtain (2.4).

The validity of (iii) and (iv) can be checked analogously to in the proof of (i) and (ii), respectively. □

The following result is an immediate consequence of Proposition 2.17.

Proposition 2.18. *Let $p, q \in \mathbb{N}$. Let F, $F_1 \in \mathcal{P}_{p\times p,\mathbb{C}}$ and let $\widetilde{F} \in \mathcal{P}_{q\times p,\mathbb{C}}$. Then F_1 is a g.r.c.d of F and $\widetilde{F}$ if and only if $F_1^{\vee}$ is a g.l.c.d of $F^{\vee}$ and $\widetilde{F}^{\vee}$.*

In the following we establish the relation between g.r.c.ds or g.l.c.ds of matrix polynomials and that of their transformations.

Proposition 2.19. *Let $p, q \in \mathbb{N}$. Let F, $F_1 \in \mathcal{P}_{p\times p,\mathbb{C}}$.*

(i) *Let $\widetilde{F} \in \mathcal{P}_{q\times p,\mathbb{C}}$ and let $U \in \mathcal{P}_{(p+q)\times(p+q),\mathbb{C}}$ be unimodular. Let $E \in \mathcal{P}_{p\times p,\mathbb{C}}$ and $\widetilde{E} \in \mathcal{P}_{q\times p,\mathbb{C}}$ be defined as follows:*

$$\begin{pmatrix} E(z) \\ \widetilde{E}(z) \end{pmatrix} := U(z) \cdot \begin{pmatrix} F(z) \\ \widetilde{F}(z) \end{pmatrix}.$$

Then F_1 is a g.r.c.d of F and $\widetilde{F}$ if and only if F_1 is a g.r.c.d of E and $\widetilde{E}$.

(ii) *Let $\widetilde{F} \in \mathcal{P}_{p\times q,\mathbb{C}}$ and let $U \in \mathcal{P}_{(p+q)\times(p+q),\mathbb{C}}$ be unimodular. Let $E \in \mathcal{P}_{p\times p,\mathbb{C}}$ and $\widetilde{E} \in \mathcal{P}_{p\times q,\mathbb{C}}$ be defined as follows:*

$$\Big(E(z), \widetilde{E}(z)\Big) := \Big(F(z), \widetilde{F}(z)\Big) \cdot U(z).$$

Then F_1 is a g.l.c.d of F and $\widetilde{F}$ if and only if F_1 is a g.l.c.d of E and $\widetilde{E}$.

Proof. The proof for (i): Suppose that F_1 is a g.r.c.d of F and $\widetilde{F}$. An application of Proposition 2.17 reveals that there exists a unimodular $U_R \in \mathcal{P}_{(p+q)\times(p+q),\mathbb{C}}$ such that (2.4) holds. Let, for $z \in \mathbb{C}$,

$$\widetilde{U}(z) := U_R(z)(U(z))^{-1}.$$

Then, by Remark 2.5, $\widetilde{U}$ is unimodular. It follows from (2.4) that

$$\widetilde{U}(z) \cdot \begin{pmatrix} E(z) \\ \widetilde{E}(z) \end{pmatrix} = \begin{pmatrix} F_1(z) \\ 0_{q\times p} \end{pmatrix}.$$

Due to Proposition 2.17 again, we obtain that F_1 is a g.r.c.d of E and $\widetilde{E}$.

Conversely, suppose that F_1 is a g.r.c.d of E and $\widetilde{E}$. By Proposition 2.17, there exists a unimodular $U_R \in \mathcal{P}_{(p+q)\times(p+q),\mathbb{C}}$ such that

$$U_R(z) \cdot \begin{pmatrix} E(z) \\ \widetilde{E}(z) \end{pmatrix} = \begin{pmatrix} F_1(z) \\ 0_{q\times p} \end{pmatrix}. \tag{2.7}$$

Let

$$\widetilde{U}_R(z) := U_R(z)U(z).$$

Then we turn (2.7) into

$$\widetilde{U}_R(z) \cdot \begin{pmatrix} F(z) \\ \widetilde{F}(z) \end{pmatrix} = \begin{pmatrix} F_1(z) \\ 0_{q\times p} \end{pmatrix}. \tag{2.8}$$

A combination of (2.8) and the fact that $\widetilde{U}_R$ is unimodular reveals that F_1 is a g.r.c.d of F and $\widetilde{F}$.

The proof for (ii) is analogous and thus omitted here. □

Proposition 2.20. *Let $p \in \mathbb{N}$. Let F, $\widetilde{F}$, W, $\widetilde{W} \in \mathcal{P}_{p\times p,\mathbb{C}}$. Let F_1 be a g.r.c.d of F and $\widetilde{F}$ and let F_2 be a g.r.c.d of WF and $\widetilde{W}\widetilde{F}$. Then*

$$\sigma(F_1) \cup \sigma(W) \cup \sigma(\widetilde{W}) = \sigma(F_2) \cup \sigma(W) \cup \sigma(\widetilde{W}). \tag{2.9}$$

Proof. F_1 is a common right divisor of WF and $\widetilde{W}\widetilde{F}$, which implies that

$$\sigma(F_1) \subseteq \sigma(F_2). \tag{2.10}$$

In the case that $\sigma(F_2) \subseteq \sigma(W) \cup \sigma(\widetilde{W})$, by (2.10) we easily obtain (2.9). Suppose that $\sigma(F_2) \nsubseteq \sigma(W) \cup \sigma(\widetilde{W})$ and there exists a $z_0 \in \mathbb{C}$ such that $z_0 \in \sigma(F_2) \setminus \Big(\sigma(W) \cup \sigma(\widetilde{W})\Big)$. Then $\det F_2(z_0) = 0$ and then $\det(W(z_0)F(z_0)) = 0$ and $\det(\widetilde{W}(z_0)\widetilde{F}(z_0)) = 0$. Since both $W(z_0)$ and $\widetilde{W}(z_0)$ are nonsingular, $\det F(z_0) = 0$ and $\det \widetilde{F}(z_0) = 0$. So $z_0 \in \sigma(F) \cap \sigma(\widetilde{F})$, or equivalently, $z_0 \in \sigma(F_1)$, which means that

$$\sigma(F_2) \setminus (\sigma(W) \cup \sigma(\widetilde{W})) \subseteq \sigma(F_1) \setminus (\sigma(W) \cup \sigma(\widetilde{W})). \tag{2.11}$$

Then (2.9) follows from (2.10) and (2.11). □

Adopting the analogous proof as in Proposition 2.20, we have that

Proposition 2.21. *Let $p \in \mathbb{N}$. Let F, $\widetilde{F}$, W, $\widetilde{W} \in \mathcal{P}_{p\times p,\mathbb{C}}$. Let F_1 be a g.l.c.d of F and $\widetilde{F}$ and let F_2 be a g.l.c.d of FW and $\widetilde{F}\widetilde{W}$. Then*

$$\sigma(F_1) \cup \sigma(W) \cup \sigma(\widetilde{W}) = \sigma(F_2) \cup \sigma(W) \cup \sigma(\widetilde{W}). \tag{2.12}$$

We conclude this section with some facts regarding to the g.r.c.ds (resp. g.l.c.ds) of composite matrix polynomials.

Definition 2.22. Let $p, q, n \in \mathbb{N}$. Let $f \in \mathcal{P}_{1\times 1,\mathbb{C}}$ and let $F \in \mathcal{P}_{p\times q,n,\mathbb{C}}$ be defined as follows

$$F(z) = \sum_{k=0}^{n} A_{n-k} z^k,$$

where $A_k \in \mathbb{C}^{p\times q}$ for $k \in \mathbb{Z}_{0,n}$. Let $F \circ f : \mathbb{C} \to \mathbb{C}^{p\times q}$ be defined by

$$F \circ f(z) := F(f(z)) = \sum_{k=0}^{n} A_{n-k} \cdot (f(z))^k.$$

Then we call $F \circ f$ the *composite polynomial of F and f.*

Remark 2.23. Let $p, q, m, n \in \mathbb{N}$. Let $F \in \mathcal{P}_{q\times p,n,\mathbb{C}}$ such that $\sigma(F) \neq \emptyset$ and let $f \in \mathcal{P}_{1\times 1,m,\mathbb{C}}$. Then $F \circ f \in \mathcal{P}_{q\times p,mn,\mathbb{C}}$. Moreover, suppose that $\lambda \in \mathbb{C}$. Then the following statements are equivalent:

(i) $\lambda \in \sigma(F \circ f)$.

(ii) $f(\lambda) \in \sigma(F)$.

(iii) There exists a $\lambda_0 \in \sigma(F)$ such that $\lambda \in \sigma(f_{\lambda_0})$, where $f_{\lambda_0}(z) := f(z) - \lambda_0$ for $z \in \mathbb{C}$.

Remark 2.24. Let $p, q \in \mathbb{N}$ and let α, $\beta \in \mathbb{C}$. Let F, $\widetilde{F} \in \mathcal{P}_{p\times q,\mathbb{C}}$ and let $f \in \mathcal{P}_{1\times 1,\mathbb{C}}$. Then
$$(\alpha F + \beta\widetilde{F}) \circ f = \alpha \cdot (F \circ f) + \beta \cdot (\widetilde{F} \circ f).$$

Remark 2.25. Let $p, q, r \in \mathbb{N}$. Let $F \in \mathcal{P}_{p\times q,\mathbb{C}}$, $\widetilde{F} \in \mathcal{P}_{q\times r,\mathbb{C}}$ and let $f \in \mathcal{P}_{1\times 1,\mathbb{C}}$. Then
$$(F \cdot \widetilde{F}) \circ f = F \circ f \cdot \widetilde{F} \circ f.$$

Proposition 2.26. *Let $p, q \in \mathbb{N}$. Let $f \in \mathcal{P}_{1\times 1,\mathbb{C}}$ and let $F \in \mathcal{P}_{p\times p,\mathbb{C}}$ be regular.*

(i) *Let $\widetilde{F} \in \mathcal{P}_{q\times p,\mathbb{C}}$. Let $F_1 \in \mathcal{P}_{p\times p,\mathbb{C}}$. If F_1 is a g.r.c.d of F and $\widetilde{F}$, then $F_1 \circ f$ is a g.r.c.d of $F \circ g$ and $\widetilde{F} \circ f$.*

(ii) *Let $\widetilde{F} \in \mathcal{P}_{p\times q,\mathbb{C}}$. Let $F_1 \in \mathcal{P}_{p\times p,\mathbb{C}}$. If F_1 is a g.l.c.d of F and $\widetilde{F}$, then $F_1 \circ f$ is a g.l.c.d of $F \circ f$ and $\widetilde{F} \circ f$.*

Proof. Here we only give the proof for (i), since (ii) is analogously verified and thus omitted.

Suppose that F_1 is a g.r.c.d of F and $\widetilde{F}$. Then due to Proposition 2.17, there exists a unimodular $U_R \in \mathcal{P}_{(p+q)\times(p+q),\mathbb{C}}$ and a $F_1 \in \mathcal{P}_{q\times p,\mathbb{C}}$ such that (2.4) holds. Let U_R be splited into a 2 by 2 block matrix as follows:
$$U_R := \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$
where $U_{11} \in \mathcal{P}_{p\times p,\mathbb{C}}$, $U_{12} \in \mathcal{P}_{p\times q,\mathbb{C}}$, $U_{21} \in \mathcal{P}_{q\times p,\mathbb{C}}$ and $U_{22} \in \mathcal{P}_{q\times q,\mathbb{C}}$. Then the comparison of both sides of (2.4) gives that
$$U_{11}(z) \cdot F(z) + U_{12}(z) \cdot \widetilde{F}(z) = F_1(z), \tag{2.13}$$
$$U_{21}(z) \cdot F(z) + U_{22}(z) \cdot \widetilde{F}(z) = 0_{q\times p}. \tag{2.14}$$
By substituting $f(z)$ for z in the formulas (2.13) and (2.14), we obtain that
$$U_{11} \circ f(z) \cdot F \circ f(z) + U_{12} \circ f(z) \cdot \widetilde{F} \circ f(z) = F_1 \circ f(z),$$
$$U_{21} \circ f(z) \cdot F \circ f(z) + U_{22} \circ f(z) \cdot \widetilde{F} \circ f(z) = 0_{q\times p},$$
or equivalently,
$$U_R \circ f(z) \cdot \begin{pmatrix} F \circ f(z) \\ \widetilde{F} \circ f(z) \end{pmatrix} = \begin{pmatrix} F_1 \circ f(z) \\ 0_{q\times p} \end{pmatrix}.$$
On the other hand, using the fact that U_R is unimodular, apparently we have det $U_R(f(z)) \in \mathbb{C}\backslash\{0\}$, which implies that $U_R \circ f$ is unimodular. Due to Proposition 2.17 again, we get that $F_1 \circ f$ is a g.r.c.d of $F \circ f$ and $\widetilde{F} \circ f$.

$\square$

3 Matrix sequences and their connection to truncated matricial moment problems

In this chapter, we present a short introduction to moment sequences of matrix measures on the real axis and the real semiaxis, which appear in the Hamburger matrix moment problem and the Stieltjes matrix moment problem, respectively. Here, we will emphasize the importance of the moment sequence of matrix measures for the solvability of these moment problems. First, some terminologies for matrix sequences and related structured matrices are introduced.

Let $p, q \in \mathbb{N}$ and let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $\mathbb{C}_\kappa^{p\times q}$ be the set of all matrix sequences over $\mathbb{C}^{p\times q}$ indexed by $\mathbb{Z}_{0,\kappa}$, that is,

$$\mathbb{C}_\kappa^{p\times q} := \left\{ (s_j)_{j=0}^{\kappa} : s_j \in \mathbb{C}^{p\times q} \text{ for } j \in \mathbb{Z}_{0,\kappa} \right\}.$$

Let $\varphi \in \mathbb{N}_0$ and let $\mathbb{C}_{\infty,\varphi,H}^{p\times p}$ be the subset of $\mathbb{C}_\infty^{p\times p}$ given by

$$\mathbb{C}_{\infty,\varphi,H}^{p\times p} := \left\{ (s_j)_{j=0}^{\infty} \in \mathbb{C}_\infty^{p\times p} : s_j \in \mathbb{C}_H^{p\times p} \text{ for } j \in \mathbb{Z}_{0,\varphi} \right\}.$$

In other words, $\mathbb{C}_{\infty,\varphi,H}^{p\times p}$ is defined as the set of all infinite matrix sequences over $\mathbb{C}^{p\times p}$, whereby the proceeding φ-th elements are Hermitian matrices.

Let $\mathscr{S} = (s_j)_{j\in\mathbb{N}_0} \in \mathbb{C}_\infty^{p\times q}$. Let Δ be the shift homomorphism of $\mathbb{C}_\infty^{p\times q}$ which is given by

$$\Delta\mathscr{S} := (s_{j+1})_{j=0}^{\infty} \in \mathbb{C}_\infty^{p\times q}, \tag{3.1}$$

and subsequently, for each $k \in \mathbb{N}$, let

$$\Delta^{(k)}\mathscr{S} := (s_{j+k+1})_{j=0}^{\infty} \in \mathbb{C}_\infty^{p\times q}.$$

Let $k \in \mathbb{N}_0$ and let $\mathscr{S}_{\langle k\rangle}$ be the k-th truncated sequence of $\mathscr{S}$ given by

$$\mathscr{S}_{\langle k\rangle} := (s_j)_{j=0}^{k} \in \mathbb{C}_k^{p\times q}. \tag{3.2}$$

Next we introduce certain structured matrices associated with matrix sequences. Suppose that $p \in \mathbb{N}$, $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and $\mathscr{S} := (s_j)_{j=0}^{\kappa} \in \mathbb{C}_\kappa^{p\times p}$. Let $k \in \mathbb{Z}_{0,\kappa}$ and let

$\mathcal{S}_k^{\langle \mathrm{I} \rangle}(\mathscr{S})$, $\mathcal{S}_k^{\langle \mathrm{II} \rangle}(\mathscr{S})$, $\mathcal{S}_k^{\langle \mathrm{III} \rangle}(\mathscr{S})$ and $\mathcal{S}_k^{\langle \mathrm{IV} \rangle}(\mathscr{S})$ be associated with $\mathscr{S}$ as follows:

$$\mathcal{S}_k^{\langle \mathrm{I} \rangle}(\mathscr{S}) := \begin{pmatrix} s_0 & s_1 & s_2 & \cdots & s_k \\ 0_p & s_0 & s_1 & \cdots & s_{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0_p & 0_p & 0_p & \cdots & s_0 \end{pmatrix},$$

$$\mathcal{S}_k^{\langle \mathrm{II} \rangle}(\mathscr{S}) := \begin{pmatrix} s_k & s_{k-1} & \cdots & s_1 & s_0 \\ s_{k-1} & s_{k-2} & \cdots & s_0 & 0_p \\ \vdots & \vdots & & \vdots & \vdots \\ s_0 & 0_p & \cdots & 0_p & 0_p \end{pmatrix},$$

$$\mathcal{S}_k^{\langle \mathrm{III} \rangle}(\mathscr{S}) := \begin{pmatrix} s_0 & 0_p & 0_p & \cdots & 0_p \\ s_1 & s_0 & 0_p & \cdots & 0_p \\ \vdots & \vdots & \vdots & & \vdots \\ s_k & s_{k-1} & s_{k-2} & \cdots & s_0 \end{pmatrix},$$

$$\mathcal{S}_k^{\langle \mathrm{IV} \rangle}(\mathscr{S}) := \begin{pmatrix} 0_p & 0_p & \cdots & 0_p & s_0 \\ 0_p & 0_p & \cdots & s_0 & s_1 \\ \vdots & \vdots & & \vdots & \vdots \\ s_0 & s_1 & \cdots & s_{k-1} & s_k \end{pmatrix}.$$

We also associate with $\mathscr{S}$ the block Hankel matrix $\mathbf{H}_{k,j}^{\langle l \rangle}(\mathscr{S})$

$$\mathbf{H}_{k,j}^{\langle l \rangle}(\mathscr{S}) := \begin{pmatrix} s_l & s_{l+1} & s_{l+2} & \cdots & s_{l+j} \\ s_{l+1} & s_{l+2} & s_{l+3} & \cdots & s_{l+k+1} \\ \vdots & \vdots & \vdots & & \vdots \\ s_{l+k} & s_{l+k+1} & s_{l+k+2} & \cdots & s_{l+k+j} \end{pmatrix},$$

where $(j,k,l) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0$ such that $j+k+l \leq \kappa$, as well as the block row vector $\mathbf{Y}_{j,k}$ and the block column vector $\mathbf{Z}_{j,k}$ generated by $\mathscr{S}$

$$\mathbf{Y}_{j,k}(\mathscr{S}) := \begin{pmatrix} s_j \\ s_{j+1} \\ \vdots \\ s_k \end{pmatrix}, \quad \mathbf{Z}_{j,k}(\mathscr{S}) := (s_j, s_{j+1}, \ldots, s_k),$$

where $k \in \mathbb{Z}_{0,\kappa}$. For simplicity we write $\mathbf{H}_k^{\langle l \rangle}(\mathscr{S})$ for $\mathbf{H}_{k,k}^{\langle l \rangle}(\mathscr{S})$ and $\mathbf{H}_k(\mathscr{S})$ for $\mathbf{H}_{k,k}^{\langle 0 \rangle}(\mathscr{S})$.

Further suppose for $k \in \mathbb{N}_0$,

$$\mathbf{L}_{1,k}(\mathscr{S}) := s_{2k} - \mathbf{Z}_{k,2k-1}(\mathscr{S}) \left(\mathbf{H}_{k-1}(\mathscr{S})\right)^{\dagger} \mathbf{Y}_{k,2k-1}(\mathscr{S}).$$

For $k \in \mathbb{N}$, $\mathbf{L}_{1,k}(\mathscr{S})$ is the Schur complement $\mathbf{H}_k(\mathscr{S})/\mathbf{H}_{k-1}(\mathscr{S})$ of $\mathbf{H}_{k-1}(\mathscr{S})$ in $\mathbf{H}_k(\mathscr{S})$.

Now our purpose is to introduce a unified terminology for some matricial moment problems and then to specify two basic types according to the two differently chosen Ω.

Let Ω be a non-empty Borelian subset of $\mathbb{R}$. Further, let $\mathfrak{B}_\Omega$ be the σ-algebra of all Borelian subsets of Ω, and let $\mathcal{M}_{\geq}^{p}(\Omega)$ be the set of all nonnegative Hermitian $p \times p$ measures on $(\Omega, \mathfrak{B}_\Omega)$. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $\mathcal{M}_{\geq,\kappa}^{p}(\Omega)$ be the set of all $\tau \in \mathcal{M}_{\geq}^{p}(\Omega)$ such that the integral

$$s_j^{(\tau)} := \int_\Omega u^j \tau(\mathrm{d}u)$$

exists for each $j \in \mathbb{Z}_{0,\kappa}$. Then we consider the following truncated matrix moment problems:

Problem $\mathbf{M}[\Omega; (s_j)_{j=0}^{n}, =]$. Let $n \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{n} \in \mathbb{C}_n^{p\times p}$. Describe the set $\mathcal{M}_{\geq}^{p}[\Omega; (s_j)_{j=0}^{n}, =]$ of all $\tau \in \mathcal{M}_{\geq,\kappa}^{p}(\Omega)$ for which $s_j^{(\tau)} = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0,n}$.

Problem $\mathbf{M}[\Omega; (s_j)_{j=0}^{n}, \leq]$. Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{n} \in \mathbb{C}_n^{p\times p}$. Describe the set $\mathcal{M}_{\geq}^{p}[\Omega; (s_j)_{j=0}^{n}, \leq]$ of all $\tau \in \mathcal{M}_{\geq,\kappa}^{p}(\Omega)$ for which $s_n^{(\tau)} - s_n$ is nonnegative Hermitian and, in the case $n > 0$, moreover, $s_j^{(\tau)} = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0,n-1}$.

The following discussions will be divided into the cases when Ω is chosen to be $\mathbb{R}$ and $[0, \infty)$, respectively. First we consider the case that $\Omega = \mathbb{R}$, which is connected to matrical versions of the classical truncated Hamburger moment problem. This matrix moment problem was intensely investigated in the literature(see, e.g. Kovalishina [55], Dym [22], Bolotnikov [7], Chen and Hu [13, 14], Fritzsche, Kirstein, Mädler [30, 33] and their joint work with Dyukarev and Thiele [28], and [46]). In order to explain some criteria of the solvability of the truncated Hamburger matrix moment problems, we will introduce some classes of matrix sequences below.

Definition 3.1. Let $m \in \mathbb{N}_0 \cup \{\infty\}$ and let $\mathscr{S} \in \mathbb{C}_{2m}^{p\times p}$. Then $\mathscr{S}$ is called *Hankel nonnegative definite* (resp. *Hankel positive definite*) if the block Hankel matrix $\mathbf{H}_m(\mathscr{S})$ is nonnegative Hermitian (resp. positive Hermitian). We denote by $\mathcal{H}_{p,2m}^{\geq}$ (resp. $\mathcal{H}_{p,2m}^{>}$) the set of all Hankel nonnegative definite (resp. Hankel positive definite) sequences $(s_j)_{j=0}^{2m}$ of $\mathbb{C}^{p\times p}$. A sequence $(s_j)_{j=0}^{2m}$ is called *Hankel nonnegative definite extendable* if there exist matrices s_{2m+1} and s_{2m+2} from $\mathbb{C}^{p\times p}$ such that $(s_j)_{j=0}^{2m+2} \in \mathcal{H}_{p,2m+2}^{\geq}$. A sequence $(s_j)_{j=0}^{2m+1}$ is called *Hankel nonnegative definite extendable* if there exists a matrix $s_{2m+2} \in \mathbb{C}_{2m+2}^{p\times p}$ such that $(s_j)_{j=0}^{2m+2} \in \mathcal{H}_{p,2m+2}^{\geq}$. For $n \in \mathbb{N}$ we denote by $\mathcal{H}_{p,n}^{\geq,e}$ the set of all Hankel nonnegative definite extendable sequences $\mathscr{S} \in \mathbb{C}_n^{p\times p}$.

The solvability of the truncated Hamburger matrix moment problems will be described in the following two theorems.

Theorem 3.2. *(see [30, Theorem 6.6] or [13, Theorem 3.1]) Let $m \in \mathbb{N}$ and let $(s_j)_{j=0}^{2m} \in \mathbb{C}_{2m}^{p\times p}$. Then $\mathcal{M}_{\geq}^{p}[\mathbb{R}; (s_j)_{j=0}^{2m}, =] \neq \emptyset$ if and only if $(s_j)_{j=0}^{2m} \in \mathcal{H}_{p,2m}^{\geq,e}$.*

Theorem 3.3. *(see [13, Theorem 3.2]) Let* $m \in \mathbb{N}$ *and let* $(s_j)_{j=0}^{2m} \in \mathbb{C}_{2m}^{p\times p}$. *Then* $\mathcal{M}_{\geq}^{p}[\mathbb{R}; (s_j)_{j=0}^{2m}, \leq] \neq \emptyset$ *if and only if* $(s_j)_{j=0}^{2m} \in \mathcal{H}_{p,2m}^{\geq}$.

Next our attention turns to the case of $\Omega = [0, \infty)$. In this case, it is referred here to the truncated Stieltjes matrix moment problem. For the intense study of this matrix moment problem, we refer the reader to Adamyan and Tkachenko [1, 2], Andô [3], Bolotnikov [6, 7, 8], Bolotnikov and Sakhnovich [9], Chen and Hu [15], Chen and Li [16], Dyukarev [24, 27], Hu and Chen [45]. In the following we introduce some matrix sequences related to the truncated Stieltjes matrix moment problem.

Definition 3.4. Let $n \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{n} \in \mathbb{C}_n^{p\times p}$. Then $(s_j)_{j=0}^{n}$ is called *Stieltjes nonnegative definite* (resp. *Stieltjes positive definite*) if $(s_j)_{j=0}^{2[\frac{n}{2}]} \in \mathcal{H}_{p,2[\frac{n}{2}]}^{\geq}$ and $(s_{j+1})_{j=0}^{2[\frac{n-1}{2}]} \in \mathcal{H}_{p,2[\frac{n-1}{2}]}^{\geq}$ (resp. $(s_j)_{j=0}^{2[\frac{n}{2}]} \in \mathcal{H}_{p,2[\frac{n}{2}]}^{>}$ and $(s_{j+1})_{j=0}^{2[\frac{n-1}{2}]} \in \mathcal{H}_{p,2[\frac{n-1}{2}]}^{>}$). We denote by $\mathcal{K}_{p,n}^{\geq}$ (resp. $\mathcal{K}_{p,n}^{>}$) the set of all Stieltjes nonnegative definite (resp. Stieltjes positive definite) sequences $(s_j)_{j=0}^{n}$ of $\mathbb{C}^{p\times p}$. Let $n \in \mathbb{N}$ and $(s_j)_{j=0}^{n} \in \mathbb{C}_n^{p\times p}$. Then $(s_j)_{j=0}^{n}$ is called *Stieltjes nonnegative definite extendable* if there exists a complex $p \times p$ matrix s_{n+1} such that $(s_j)_{j=0}^{n+1} \in \mathcal{K}_{p,n+1}^{\geq}$. For $n \in \mathbb{N}$ we denote by $\mathcal{K}_{p,n}^{\geq,e}$ the set of all Stieltjes nonnegative definite extendable sequences $(s_j)_{j=0}^{n}$ from $\mathbb{C}^{p\times p}$.

Remark 3.5. [27, Remark 4.5] If $p, n \in \mathbb{N}$, then $\mathcal{K}_{p,n}^{>} \subseteq \mathcal{K}_{p,n}^{\geq,e}$.

Remark 3.6. Let $m \in \mathbb{N}$ and let $(s_j)_{j=0}^{2m}$ be from $\mathbb{C}^{p\times p}$. By Definitions 3.1 and 3.4, we have

(i) $(s_j)_{j=0}^{2m} \in \mathcal{K}_{p,2m}^{\geq,e}$ if and only if $(s_j)_{j=0}^{2m} \in \mathcal{H}_{p,2m}^{\geq}$ and $(s_{j+1})_{j=0}^{2m-1} \in \mathcal{H}_{p,2m-1}^{\geq,e}$.

(ii) $(s_j)_{j=0}^{2m-1} \in \mathcal{K}_{p,2m-1}^{\geq,e}$, if and only if $(s_{j+1})_{j=0}^{2m-2} \in \mathcal{H}_{p,2m-2}^{\geq}$ and $(s_j)_{j=0}^{2m-1} \in \mathcal{H}_{p,2m-1}^{\geq,e}$.

Now we recall the well known solvability criteria of the truncated Stieltjes matrix moment problems (see e.g., Lemma 1.7 of [7], Lemma 1.2 of [15], Theorem 1.4 of [27], Theorem 1.1 of [45]).

Theorem 3.7. *Let* $n \in \mathbb{N}$ *and let* $(s_j)_{j=0}^{n} \in \mathbb{C}_n^{p\times p}$. *Then* $\mathcal{M}_{\geq}^{p}[[0,\infty); (s_j)_{j=0}^{n}, \leq] \neq \emptyset$ *if and only if* $(s_j)_{j=0}^{n} \in \mathcal{K}_{p,n}^{\geq}$.

Theorem 3.8. *Let* $n \in \mathbb{N}$ *and let* $(s_j)_{j=0}^{n} \in \mathbb{C}_n^{p\times p}$. *Then* $\mathcal{M}_{\geq}^{p}[[0,\infty); (s_j)_{j=0}^{n}, =] \neq \emptyset$ *if and only if* $(s_j)_{j=0}^{n} \in \mathcal{K}_{p,n}^{\geq,e}$.

4 Matrix fraction description and some related topics

Up until the 1960s there was little study of the internal state of linear systems in the literature. The famous state space theory, as initiated by Kalman [50, 51, 52] and Livšic [64], emerged to fill this vacancy and, moreover, provided us with the important language for the study of multivariable linear systems. Later, aware of the lack of transparency in the state space approach, such pioneers as Rosenbrock [69], Popov [67], Wolovich [71] et al. developed a better approach called “polynomial systems theory”.

As an important tool in this theory, matrix fraction description inherits the power of scalar transfer function and, moreover, offers a better understanding of the connection between the “polynomial systems theory” and the state space theory. Hence, this approach obtains a natural generalization from the single input-single output transfer function to multiple input-multiple output cases.

In this chapter we look more closely at the previously mentioned matrix fraction description. More specifically, we choose two dual types of matrix fraction descriptions, namely normalized left (resp. right) matrix fraction description (for short NLFD (resp. NRFD)), for the study of strictly proper transfer function matrices G.

In Section 4.1 we check the existence of NRFD (resp. NLFD) for G and give a realization from the SMP of G to its NRFD (resp. NLFD). As a result, by virtue of the SMP of G we characterize the case that G is a Hermitian strictly proper transfer function matrix.

Section 4.2 subsequently considers a particular sequence of Hermitian transfer function matrices $(G_k)_{k=1}^m$, which is connected with an infinite matrix sequence $\mathscr{S}$ such that $\mathscr{S}_{\langle 2k-1\rangle}$ is the SMP of G_k. We are furthermore able to find out some correspondences between NRFDs (resp. NLFDs) of $(G_k)_{k=1}^m$ and a monic orthogonal system of matrix polynomials with respect to $\mathscr{S}$.

4.1 Realization of Matrix fraction description from Markov parameters

In order to familiarize the reader with transfer function matrices and their matrix fraction description, we begin this section by recalling the role of transfer function in the single-input single-output (SISO) continuous-time linear systems.

The dynamics of a linear continuous-time system (A, B, C, D) with zero initial state is described by the following system of equations

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

with $x(0) = 0$, where x is the state vector, u is the input, y is the output and A, B, C and D are some square matrices. The related transfer function

$$G(s) = C(sI - A)^{-1}B + D = \frac{C\mathrm{Adj}(sI - A)B}{\det(sI - A)} + D$$

is the ratio between the Laplace transform of output y and the Laplace transform of input x.

Definition 4.1. A map $g : \mathbb{C} \to \mathbb{C}$ is called a *transfer function* if g is a rational function that is represented by

$$g(z) = n(z)(d(z))^{-1},$$

where n, $d \in \mathcal{P}_{1\times 1,\mathbb{C}}$. Here we denote by $N(g)$ the set of all poles of g. More particularly, g is called a *proper* (*strictly proper*, resp.) transfer function if deg $n \leq$ deg d (deg $n <$ deg d, resp.).

We pay attention to some multivariable analogs for SISO systems. For the comprehensive study of multiple-input multiple-output linear system, we refer the reader to the monographs by T. Kailath [49] and C.T. Chen [12].

Definition 4.2. A map $G : \mathbb{C} \to \mathbb{C}^{p\times q}$ is called a *transfer function matrix* if G is a $p \times q$ matrix of transfer functions $G(z) := (g_{jk}(z))_{j,k=1}^{p,q}$, where $g_{jk}(z)$ is a rational transfer function that relates input of actuator j to output of sensor k. More specially,

(i) G is called *strictly proper* (*proper*, resp.) if for each $j \in \mathbb{Z}_{1,p}$ and $k \in \mathbb{Z}_{1,q}$, g_{jk} is a strictly proper (proper, resp.) rational transfer function.

(ii) G is called *unitary* if for $z \in \mathbb{C}$,

$$G(z) \cdot (G(\bar{z}))^* \equiv I_p,$$
$$(G(\bar{z}))^* \cdot G(z) \equiv I_q.$$

(iii) G is called *Hermitian* if for $z \in \mathbb{C}$,

$$G(z) \equiv (G(\bar{z}))^*.$$

(iv) G is called *para-Hermitian* if for $z \in \mathbb{C}$,

$$G(z) \equiv (G(-\bar{z}))^*.$$

We denote by $\mathcal{T}_{p,q}$ (resp. $\mathcal{T}^{\diamond}_{p,q}$) the set of all transfer function matrices (resp. strictly proper transfer function matrices) of size $p \times q$.

Remark 4.3. Let $p \in \mathbb{N}$ and let $G \in \mathcal{T}_{p,p}$. Then G is unitary if and only if $G^{\vee}$ is unitary.

Definition 4.4. Let $p, q \in \mathbb{N}$ and let $G \in \mathcal{T}_{p,q}$. A pair $(N, D) \in \mathcal{P}_{q\times q,\mathbb{C}} \times \mathcal{P}_{p\times q,\mathbb{C}}$ is called a *right matrix fraction description* (RFD for short) for G if

$$G(z) = N(z) \cdot (D(z))^{-1}, \quad z \in \mathbb{C}. \tag{4.1}$$

Definition 4.5. Let $p, q \in \mathbb{N}$ and let $G \in \mathcal{T}_{p,q}$. A pair $(N, D) \in \mathcal{P}_{p\times q,\mathbb{C}} \times \mathcal{P}_{p\times p,\mathbb{C}}$ is called a *left matrix fraction description* (LFD for short) for G if

$$G(z) = (D(z))^{-1} N(z), \quad z \in \mathbb{C}. \tag{4.2}$$

Definition 4.6. Let $p \in \mathbb{N}$.

(i) Let $G \in \mathcal{T}_{p,p}$ and let (N, D) be an RFD for G. Then (N, D) is called *irreducible* if N and D are right coprime.

(ii) Let $G \in \mathcal{T}_{p,p}$ and let (N, D) be a LFD for G. Then (N, D) is called *irreducible* if N and D are left coprime.

Proposition 4.7. *[12, Theorem 7.8] Let $p \in \mathbb{N}$.*

(i) *Let $G \in \mathcal{T}_{p,p}$ and let (N, D) be an irreducible RFD for G. Suppose that D is column reduced. Then $G \in \mathcal{T}^{\diamond}_{p,p}$ if and only if*

$$\deg_k N < \deg_k D, \quad \forall k \in \mathbb{Z}_{1,p}.$$

(ii) *Let $G \in \mathcal{T}_{p,p}$ and let (N, D) be an irreducible LFD for G. Suppose that D is column reduced. Then $G \in \mathcal{T}^{\diamond}_{p,p}$ if and only if*

$$\deg_k N < \deg_k D, \quad \forall k \in \mathbb{Z}_{1,p}.$$

Next we will seek the connection between strictly proper transfer function matrices and normalized matrix fraction descriptions. The latter are our main consideration in this chapter as important types of matrix fraction description.

Definition 4.8. Let $p, q \in \mathbb{N}$.

(i) Let $G \in \mathcal{T}_{p,q}$ and let (N, D) be an RFD for G. (N, D) is called a *strictly proper-type right matrix fraction description* for G if $\deg N < \deg D$. Moreover, (N, D) is called an *normalized right matrix fraction description* (NRFD for short) for G if (N, D) is a strictly proper-type right matrix fraction description for G and D is monic.

(ii) Let $G \in \mathcal{T}_{p,q}$ and let (N, D) be a LFD for G. (N, D) is called a *strictly proper-type left matrix fraction description* for G if deg $N <$ deg D. Moreover, (N, D) is called an *normalized left matrix fraction description* (NLFD for short) for G if (N, D) is a strictly proper-type left matrix fraction description for G and D is monic.

Given a transfer function matrix $G \in \mathcal{T}^{\diamond}_{p,p}$, we can find an NRFD and an NLFD of G.

Proposition 4.9. *Let $p \in \mathbb{N}$ and let $G \in \mathcal{T}_{p,p}$. Then the following statements are equivalent:*

(i) $G \in \mathcal{T}^{\diamond}_{p,p}$.

(ii) *There exists an NRFD for G.*

(iii) *There exists an NLFD for G.*

Proof. The proof for "(i) $\Longrightarrow$ (ii) and (iii)": Let $G(z) := (g_{jk}(z))_{j,k=1}^{p} \in \mathcal{T}^{\diamond}_{p,p}$, where

$$g_{jk}(z) := \frac{n_{jk}(z)}{d_{jk}(z)}, \quad \forall j, k \in \mathbb{Z}_{1,p}.$$

is a strictly proper transfer function (deg n_{jk} <deg d_{jk}). Further let d be the monic least common multiple of g_{11}, g_{12}, $\ldots$, g_{pp} such that the following equations

$$d(z) = g_{jk}(z) r_{jk}(z), \quad \forall j, k \in \mathbb{Z}_{1,p}$$

hold, where $r_{jk} \in \mathcal{P}_{1\times 1,\mathbb{C}}$ for j, $k \in \mathbb{Z}_{1,p}$. Then we construct two matrix polynomials $N(z) := (n_{jk}(z) r_{jk}(z))_{j,k=1}^{p}$ and $D(z) := d(z) I_p$ for $z \in \mathbb{C}$. So (N, D) is an NRFD for G and an NLFD for G.

The proof for "(ii) $\Longrightarrow$ (i)": Suppose that (N, D) is an NRFD for G and $m :=$deg D. Further suppose that for $z \in \mathbb{C}$, $N(z) := (n_{jk}(z))_{j,k=1}^{p}$ and, for $j, k \in \mathbb{Z}_{1,p}$, $d_{jk}(z)$ is the (j, k) minor of $D(z)$. Then

$$\begin{aligned} G(z) =& N(z) D^{-1}(z) = N(z) \mathrm{Adj}(D(z)) \left(\det(D(z))\right)^{-1} \\ =& \left(\frac{\widetilde{g}_{jk}(z)}{\det(D(z))} \right)_{j,k=1}^{p}, \end{aligned}$$

where

$$\widetilde{g}_{jk}(z) := \sum_{l=1}^{p} (-1)^{l+k} n_{jl}(z) d_{kl}(z), \quad z \in \mathbb{C}.$$

Since

$$\begin{aligned} \deg\, \widetilde{g}_{jk} \leq& \max_{l \in \mathbb{Z}_{1,p}} \{\deg\, n_{jl} + \deg\, d_{kl}\} \\ <& m + (m-1)p \leq mp = \deg\, \det(D(z)), \end{aligned}$$

we hence obtain $G \in \mathcal{T}^{\diamond}_{p,p}$.

The proof for "(iii) $\Longrightarrow$ (i)" is analogous to that for "(ii) $\Longrightarrow$ (i)" and therefore omitted here. $\square$

Now we are going to recall the definition of the Markov parameters of the strictly proper transfer function matrices.

Let $p, q \in \mathbb{N}$ and let $G := (g_{jk}(z))_{j,k=1}^{p,q} \in \mathcal{T}_{p,q}^{\diamond}$. For every $j \in \mathbb{Z}_{1,p}$ and $k \in \mathbb{Z}_{1,q}$, assume that g_{jk} is represented by the following Laurent series

$$g_{jk}(z) = \sum_{l=-1}^{\infty} z^{-(l+1)} s_l^{\langle j,k \rangle}$$

for each $z \in \mathbb{C}$ and $|z| > \max_{\lambda \in N(g_{jk})} |\lambda|$, where $s_l^{\langle j,k \rangle} \in \mathbb{C}$ for $l \in \mathbb{Z}_{-1,\infty}$. Let $s_l := (s_l^{\langle j,k \rangle})_{j,k=1}^{p,q}$ for $l \in \mathbb{Z}_{-1,\infty}$ and let $\mathscr{S} := (s_l)_{l=0}^{\infty} \in \mathbb{C}_{\infty}^{p\times p}$. Accordingly, G is represented by the following Laurent series

$$G(z) = \sum_{l=-1}^{\infty} z^{-(l+1)} s_l$$

for each $z \in \mathbb{C}$ and $|z| > \max_{\substack{j \in \mathbb{Z}_{1,p} \\ k \in \mathbb{Z}_{1,m}}} \max_{\lambda \in N(g_{jk})} |\lambda|$. In this case, we call $\mathscr{S}$ is the *extended sequence of Markov parameters* (or short *SMP*) of G. Moreover, let $k \in \mathbb{N}_0$. Then we call $\mathscr{S}_{\langle k \rangle}$ the *k-th SMP of G* if $\mathscr{S}$ is the SMP of G.

Definition 4.10. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic as in (2.1). Then the *first companion matrix* $\mathbf{C}_F^{\langle 1 \rangle}$ of F is defined via

$$\mathbf{C}_F^{\langle 1 \rangle} := \begin{pmatrix} 0_p & I_p & \cdots & 0_p \\ \vdots & \vdots & \ddots & \vdots \\ 0_p & 0_p & \cdots & I_p \\ -A_n & -A_{n-1} & \cdots & -A_1 \end{pmatrix} \in \mathbb{C}^{pn\times pn}$$

and, analogously, the *second companion matrix* $\mathbf{C}_F^{\langle 2 \rangle}$ of F is defined via

$$\mathbf{C}_F^{\langle 2 \rangle} := \begin{pmatrix} 0_p & \cdots & 0_p & -A_n \\ I_p & \cdots & 0_p & -A_{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0_p & \cdots & I_p & -A_1 \end{pmatrix} \in \mathbb{C}^{pn\times pn}.$$

Starting from a given sequence of complex $p \times p$ matrices, we construct a transformation of matrix polynomials in the following way.

Definition 4.11. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $n \in Z_{0,\kappa}$. Let $\mathscr{S} \in \mathbb{C}_{\kappa}^{p\times p}$ and let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be as in (2.1). Let $F^{\langle \mathscr{S} \rangle} : \mathbb{C} \to \mathbb{C}^{p\times p}$ be defined by

$$F^{\langle \mathscr{S} \rangle}(z) := \begin{cases} 0_p, & n = 0, \\ (A_{n-1}, A_{n-2}, \ldots, A_0)\mathcal{S}_{n-1}^{\langle \mathrm{III} \rangle}(\mathscr{S}) \begin{pmatrix} I_p \\ zI_p \\ \vdots \\ z^{n-1} I_p \end{pmatrix}, & n \in \mathbb{N}. \end{cases}$$

We call $F^{\langle \mathscr{S} \rangle}$ *left $\mathscr{S}$-associated with respect to F.*

Further let $F^{\{\mathscr{S}\}} : \mathbb{C} \to \mathbb{C}^{p\times p}$ be defined by

$$F^{\{\mathscr{S}\}}(z) := \begin{cases} 0_p, & m = 0, \\ (I_p, zI_p, \dots, z^{n-1}I_p)\mathcal{S}_{n-1}^{\langle \mathrm{I} \rangle}(\mathscr{S}) \begin{pmatrix} A_{n-1} \\ A_{n-2} \\ \vdots \\ A_0 \end{pmatrix}, & n \in \mathbb{N}. \end{cases}$$

We call $F^{\{\mathscr{S}\}}$ *right $\mathscr{S}$-associated with respect to F.*

Remark 4.12. Let $n \in \mathbb{N}$. Let $\mathscr{S} \in \mathbb{C}^{p\times p}_{n-1,\infty}$ and let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$. Then $\widetilde{F}$ is left $\mathscr{S}$-associated with respect to F if and only if $\widetilde{F}^{\vee}$ is right $\mathscr{S}$-associated with respect to $F^{\vee}$.

In the following we concerned with the realization from the SMPs of strictly proper transfer function matrices to their NRFDs (resp. NLFDs).

Proposition 4.13. *Let $p, q \in \mathbb{N}$ and let $G \in \mathcal{T}^{\diamond}_{p,p}$. Let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty}$ be the SMP of G.*

(i) *Suppose that there exists an $m \in \mathbb{N}$ and a monic matrix polynomial $D \in \mathcal{P}_{p\times p,m,\mathbb{C}}$ such that for $l \in \mathbb{Z}_{1,\infty}$,*

$$\mathbf{H}_m^{\langle l \rangle}(\mathscr{S}) = \mathbf{H}_m(\mathscr{S})(\mathbf{C}_D^{\langle 2 \rangle})^l \tag{4.3}$$

and suppose that N is right $\mathscr{S}$-associated with D. Then (N, D) is an NRFD for G.

(ii) *Conversely, suppose that (N, D) is an NRFD for G and m :=deg D. Then* (4.3) *holds for $l \in \mathbb{Z}_{1,\infty}$ and N is right $\mathscr{S}$-associated with D.*

Proof. The proof of (i): Suppose that there exists an $m \in \mathbb{N}$ and a monic matrix polynomial $D \in \mathcal{P}_{p\times p,m,\mathbb{C}}$ such that (4.3) holds for $l \in \mathbb{Z}_{1,\infty}$. Let $D(z) := \sum_{j=0}^{m} z^{m-j}D_j \in \mathcal{P}_{p\times p,m,\mathbb{C}}$, where $D_0 := I_p$. We rewrite (4.3), by comparing its both sides, into the following form:

$$\begin{pmatrix} s_m & \cdots & s_0 & 0_p & \cdots & 0_p \\ s_{m+1} & \cdots & s_1 & s_0 & \cdots & 0_p \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ s_{m+l} & \cdots & s_l & s_{l-1} & \cdots & s_0 \end{pmatrix} \begin{pmatrix} I_p \\ D_1 \\ \vdots \\ D_m \\ 0_{lp\times p} \end{pmatrix} = 0_{(l+1)p\times p}. \tag{4.4}$$

Let N be right $\mathscr{S}$-associated with D and admit the following representation $N(z) :=$

$\sum_{j=0}^{m-1} z^{m-1-j} N_j$. It follows from (4.4) that

$$\begin{pmatrix} N_0 \\ N_1 \\ \vdots \\ N_{m-1} \\ 0_{(k-m+2)p\times p} \end{pmatrix} = \mathcal{S}_{k+1}^{\langle \mathrm{III}\rangle}(\mathscr{S}) \begin{pmatrix} I_p \\ D_1 \\ \vdots \\ D_m \\ 0_{(k-m+1)p\times p} \end{pmatrix}, \tag{4.5}$$

for each $k \in \mathbb{Z}_{0,n-1}$, which implies that

$$N(z) = \left(\sum_{j=0}^{\infty} z^{-(j+1)} s_j \right) \cdot D(z). \tag{4.6}$$

Right multiplication of both sides in (4.6) by D^{-1} yields that

$$N(z)(D(z))^{-1} = \sum_{j=0}^{\infty} z^{-(j+1)} s_j \tag{4.7}$$

holds for each each $z \in \mathbb{C}$ and $|z| >$max$\{|\lambda| : \lambda \in \sigma(D)\}$. Hence $G(z) = N(z)(D(z))^{-1}$ and then (N, D) is an NRFD for G.

The proof of (ii): Conversely, suppose that (N, D) is an NRFD for G. Then (4.7) holds and, using the right multiplication of both sides in (4.7) by D, the equality(4.6) is verified. Comparing the both sides of (4.6) term by term, we have (4.5) or, equivalently, (4.3) and the fact that N is right $\mathscr{S}$-associated with D. □

By adopting analogous approach as in Proposition 4.13, we have

Proposition 4.14. *Let $p \in \mathbb{N}$ and let $G \in \mathcal{T}_{p,p}^{\diamond}$. Let $\mathscr{S} \in \mathbb{C}_{\infty}^{p\times p}$ be the SMP of G.*

(i) *Suppose that there exists an $m \in \mathbb{N}$ and a monic $D \in \mathcal{P}_{p\times p,m,\mathbb{C}}$ such that for $l \in \mathbb{N}$,*
$$\mathbf{H}_m^{\langle l\rangle}(\mathscr{S}) = (\mathbf{C}_D^{\langle 1\rangle})^l \mathbf{H}_m(\mathscr{S}). \tag{4.8}$$
Then (N, D) is an NLFD for G, where N is left $\mathscr{S}$-associated with D.

(ii) *Conversely, suppose that (N, D) is an NLFD for G. Then* (4.8) *holds for $l \in \mathbb{N}$ and N is left $\mathscr{S}$-associated with D.*

Next we point out the connection between Hermitian strictly transfer function matrices and their truncated Markov parameters.

Proposition 4.15. *Let p, $n \in \mathbb{N}$. Let $G \in \mathcal{T}_{p,p}^{\diamond}$ and let (N, D) be an NRFD for G. Let $\mathscr{S} \in \mathbb{C}_{\infty}^{p\times p}$ be the SMP of G. Then the following statements are equivalent:*

(i) *$\mathscr{S} \in \mathbb{C}_{\infty,n,H}^{p\times p}$, where $n :=$*deg *$N+$*deg *D.*

(ii) *G is Hermitian.*

(iii) $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,\infty,H}$.

Proof. The implication "(i)$\Longrightarrow$ (ii)": Let $\mathscr{S} := (s_j)_{j\in\mathbb{N}_0}$. Let $m :=$deg D, $l :=$deg N ($m+l=n$, $m>l$) and let

$$D(z) := D_m + D_{m-1}z + \cdots + D_1 z^{m-1} + D_0 z^m,$$
$$N(z) := N_l + N_{l-1}z + \cdots + N_1 z^{l-1} + N_0 z^l.$$

Then applying (ii) of Proposition 4.13 gives that

$$\begin{pmatrix} 0_{(m-l)p\times p} \\ N_l \\ \vdots \\ N_0 \end{pmatrix} = \mathcal{S}_m^{\langle \mathrm{I}\rangle}(\Delta^{(m-l-1)}\mathscr{S}) \begin{pmatrix} D_m \\ D_{m-1} \\ \vdots \\ D_0 \end{pmatrix}. \tag{4.9}$$

and for $l \in \mathbb{Z}_{1,\infty}$,

$$\mathbf{H}_m^{\langle l\rangle}(\mathscr{S}) = \mathbf{H}_m(\mathscr{S})(\mathbf{C}_D^{\langle 2\rangle})^l. \tag{4.10}$$

Let $G \in \mathcal{P}_{p\times p,m+l}$ be given by

$$G(z) := D^{\vee}(z)N(z).$$

Suppose that G has the following form:

$$G(z) =: R_0 + R_1 z + \cdots + R_{m+l}z^{m+l}.$$

Assume further that for $k \in \mathbb{Z}_{1,m+l}$, $E_k := (\delta_{t,k-j}I_p)_{t,j=0}^k \in \mathbb{C}^{jp\times jp}$, where for each $t,\ j \in \mathbb{Z}_{0,j}$,

$$\delta_{t,j} := \begin{cases} 1, & \text{if } t=j, \\ 0, & \text{if } t\neq j, \end{cases}$$

is the Kronecker delta function and for $k \in \mathbb{Z}_{0,m+l}$,

$$\mathbf{W}_k := \begin{cases} \begin{pmatrix} 0_{(k-l)p\times(k-l)p} & 0_{(k-l)p\times(m+l+1-k)p} \\ 0_{(m+l+1-k)p\times(k-l)p} & E_{m+l-k} \end{pmatrix}, & \text{if } k \in \mathbb{Z}_{l+1,l+m}, \\ E_m, & \text{if } k=l, \\ \begin{pmatrix} E_{m+k-l} & 0_{(m+k+1-l)p\times(l-k)p} \\ 0_{(l-k)p\times(m+k+1-l)p} & 0_{(l-k)p\times(l-k)p} \end{pmatrix}, & \text{if } k \in \mathbb{Z}_{0,l-1}. \end{cases}$$

Then for $k \in \mathbb{Z}_{0,n}$,

$$\begin{aligned} R_k &= \begin{pmatrix} D_m^* & \cdots & D_0^* \end{pmatrix} \mathbf{W}_k \begin{pmatrix} 0_{(m-l)p\times p} \\ N_l \\ \vdots \\ N_0 \end{pmatrix} \\ &= \begin{pmatrix} D_m^* & \cdots & D_0^* \end{pmatrix} \mathbf{W}_k \mathcal{S}_m^{\langle \mathrm{I}\rangle}(\Delta^{(m-l-1)}\mathscr{S}) \begin{pmatrix} D_m \\ \vdots \\ D_0 \end{pmatrix} \end{aligned} \tag{4.11}$$

If $k = l$, then by (4.11),

$$R_k = \begin{pmatrix} D_m^* & \cdots & D_0^* \end{pmatrix} \mathcal{S}_m^{\langle \mathrm{IV} \rangle}(\Delta^{(m-l-1)}\mathscr{S}) \begin{pmatrix} D_m \\ \vdots \\ D_0 \end{pmatrix} = R_k^*.$$

If $k \in \mathbb{Z}_{l+1,n}$, then by (4.11),

$$\begin{aligned} R_k &= \begin{pmatrix} D_m^* & \cdots & D_0^* \end{pmatrix} \begin{pmatrix} 0_{(k-l)p\times(k-l)p} & 0_{(k-l)p\times(n+1-k)p} \\ 0_{(n+1-k)p\times(k-l)p} & \mathcal{S}_{n-k}^{\langle \mathrm{IV} \rangle}(\Delta^{(m-l-1)}\mathscr{S}) \end{pmatrix} \begin{pmatrix} D_m \\ \vdots \\ D_0 \end{pmatrix} \\ &= R_k^*. \end{aligned}$$

If $k \in \mathbb{Z}_{0,l-1}$, then

$$\begin{aligned} R_k &= \begin{pmatrix} D_m^* & \cdots & D_0^* \end{pmatrix} \begin{pmatrix} 0_p & \cdots & s_{m-l-1} & \cdots & s_{m-k-1} \\ \vdots & \iddots & \vdots & & \vdots \\ s_{m-l-1} & \cdots & s_{2m+k-2l-1} & \cdots & s_{2m-l-1} \\ 0_p & \cdots & 0_p & \cdots & 0_p \\ \vdots & & \vdots & & \vdots \\ 0_p & \cdots & 0_p & \cdots & 0_p \end{pmatrix} \begin{pmatrix} D_m \\ \vdots \\ D_0 \end{pmatrix} \\ &= \begin{pmatrix} D_m^* & \cdots & D_0^* \end{pmatrix} \begin{pmatrix} \mathcal{S}_{m+k-l}^{\langle \mathrm{IV} \rangle}(\Delta^{(2m+k-2l-1)}\mathscr{S}) & 0_{(m+k+1-l)p\times(l-k)p} \\ 0_{(l-k)p\times(m+k+1-l)p} & -\mathbf{H}_{l-k-1}^{\langle 2m+k+1-2l \rangle}(\mathscr{S}) \end{pmatrix} \begin{pmatrix} D_m \\ \vdots \\ D_0 \end{pmatrix} \\ &= R_k^*, \end{aligned}$$

where the 1st equality is due to (4.11) and the 2nd equality is due to (4.10).

It follows that

$$G(z) = G^{\vee}(z), \quad z \in \mathbb{C},$$

or equivalently, F is Hermitian.

The implications "(ii)$\Longrightarrow$ (iii)" and "(iii)$\Longrightarrow$ (i)" are obvious.

□

4.2 The interrelation between Hermitian transfer function matrices and monic orthogonal system of matrix polynomials

With regard to Hermitian transfer function matrices, we start this section by considering the monic matricial orthogonal polynomial systems, which we will very generally consider with respect to a matrix sequence over $\mathbb{C}^{p\times p}$. These matricial orthogonal polynomial systems are originated from M.G. Kreĭn [57]. Let us recall some related definitions.

Let $P \in \mathcal{P}_{p\times p,\mathbb{C}}$ be defined as $P(z) := \sum_{j=0}^{\infty} z^j A_j$. For all $n \in \mathbb{N}_0$, let then $Z_n^{[P]} := [A_0, A_1, \ldots, A_n]$ be the row vector whose entries are A_j and $Y_n^{[P]} :=\mathrm{col}(A_j)_{j=0}^{n}$ be the column vector whose entries are A_j, where $(A_j)_{j=0}^{\infty} \in \mathbb{C}_{\infty}^{p\times p}$ is uniquely determined by the representation of P.

Definition 4.16. Let $m \in \mathbb{N} \cup \{\infty\}$ and let $\mathscr{S} \in \mathbb{C}_{\infty}^{p\times p}$. For each $k \in \mathbb{Z}_{0,m}$, let $P_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$. Then $(P_k)_{k=0}^{m}$ is called a *monic right* (resp. *left) orthogonal system of matrix polynomial* (*MROSMP* (resp. *MLOSMP*)) *of order m with respect to* $\mathscr{S}$ if for all j, $k \in \mathbb{Z}_{0,m}$ with $j \neq k$,

$$\begin{aligned}
&(Y_m^{[P_j]})^* \mathbf{H}_m(\mathscr{S}) Y_m^{[P_k]} = 0_p \\
&(\text{resp. } Z_m^{[P_j]} \mathbf{H}_m(\mathscr{S}) (Z_m^{[P_k]})^* = 0_p).
\end{aligned}$$

Definition 4.17. Let $m \in \mathbb{N}_0$. A map

$$\psi : \mathcal{P}_{p\times p,m,\mathbb{C}} \times \mathcal{P}_{p\times p,m,\mathbb{C}} \to \mathbb{C}^{p\times p}$$

is called *left* (resp. *right*) *sesquilinear* in $\mathcal{P}_{p\times p,m,\mathbb{C}}$ if for any F_1, F_2, G_1, $G_2 \in \mathcal{P}_{p\times p,m,\mathbb{C}}$ and any $a_1, a_2 \in \mathbb{C}^{p\times p}$,

$$\psi(F_1 + F_2, G_1 + G_2) = \psi(F_1, G_1) + \psi(F_1, G_2) + \psi(F_2, G_1) + \psi(F_2, G_2)$$

and

$$\begin{aligned}
\psi(a_1 \cdot F_1, a_2 \cdot G_1) &= a_1 \psi(F_1, G_1) + \psi(F_1, G_1) a_2^* \\
(\text{resp. } \psi(a_1 \cdot F_1, a_2 \cdot G_1) &= a_1^* \psi(F_1, G_1) + \psi(F_1, G_1) a_2).
\end{aligned}$$

Moreover, ψ is called *Hermitian* if additionally for any $F_1, F_2 \in \mathcal{P}_{p\times p,\mathbb{C}}$,

$$\psi(F_1, F_2) = (\psi(F_2, F_1))^*.$$

We denote by $\Psi_L(\mathcal{P}_{p\times p,m,\mathbb{C}})$ (resp. $\Psi_R(\mathcal{P}_{p\times p,m,\mathbb{C}})$) the set of all the left (resp. right) sesquilinear maps in $\mathcal{P}_{p\times p,m,\mathbb{C}}$.

Let $m \in \mathbb{N}_0$ and, for $j \in \mathbb{Z}_{0,m}$, let $\mathcal{E}_j \in \mathcal{P}_{p\times p,j,\mathbb{C}}$ be given by

$$\mathcal{E}_j(z) := z^j I_p, \quad z \in \mathbb{C}.$$

Let $\psi \in \Psi_L(\mathcal{P}_{p\times p,m,\mathbb{C}}) \cup \Psi_R(\mathcal{P}_{p\times p,m,\mathbb{C}})$. We define the Gram matrix $\mathbf{G}$ with respect to ψ by

$$\mathbf{G} = (\psi(\mathcal{E}_j, \mathcal{E}_k))_{j,k=0}^{m}. \tag{4.12}$$

Conversely, let $\mathbf{G} \in \mathbb{C}^{(m+1)p\times(m+1)p}$ be given. Then there exists a unique left (resp. right) sesquilinear map ψ such that (4.12) holds. In this case, we denote $\mathbf{G}$ by $\mathbf{G}_\psi$ and denote ψ by $\psi_{\langle \mathbf{G},L\rangle}$ (resp. $\psi_{\langle \mathbf{G},R\rangle}$).

Remark 4.18. Suppose that $\psi \in \Psi_L(\mathcal{P}_{p\times p,m,\mathbb{C}}) \cup \Psi_R(\mathcal{P}_{p\times p,m,\mathbb{C}})$ is Hermitian. Then $\mathbf{G}_\psi \in \mathbb{C}_H^{(m+1)p\times(m+1)p}$. Conversely, suppose that $\mathbf{G} \in \mathbb{C}_H^{(m+1)p\times(m+1)p}$. Then $\psi_{\langle \mathbf{G},L\rangle}$ and $\psi_{\langle \mathbf{G},R\rangle}$ are Hermitian.

Definition 4.19. Let $m \in \mathbb{N}_0$. Let F, $G \in \mathcal{P}_{p\times p,m,\mathbb{C}}$ and let $\psi \in \Psi_L(\mathcal{P}_{p\times p,m,\mathbb{C}}) \cup \Psi_R(\mathcal{P}_{p\times p,m,\mathbb{C}})$ be Hermitian. Then we say that F and G are *orthogonal with respect to* ψ if $\psi(F,G) = 0_p$. Let $F_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$ for each $k \in \mathbb{Z}_{0,m}$. Then we say that $(F_k)_{k=0}^m$ is an *orthogonalization of* $(\mathcal{E}_k)_{k=0}^m$ *with respect to* ψ if for $k, j \in \mathbb{Z}_{0,m}$ and $k \neq j$,

$$\psi(F_k, F_j) = 0_p,$$

and for $k \in \mathbb{Z}_{0,m}$,

$$\mathrm{Span}\{F_0, \dots, F_k\} = \mathrm{Span}\{\mathcal{E}_0, \dots, \mathcal{E}_k\}. \tag{4.13}$$

Remark 4.20. Let $m \in \mathbb{N}_0$ and let $P_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$ for each $k \in \mathbb{Z}_{0,m}$. Then $(P_k)_{k=0}^m$ is an orthogonalization of $(\mathcal{E}_k)_{k=0}^m$ with respect to ψ if and only if for $k \in \mathbb{Z}_{1,m}$ and $j \in \mathbb{Z}_{0,k-1}$,

$$\psi(P_k, \mathcal{E}_j) = 0_p.$$

Let $m \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}$. In what follows, we concentrate on MLOSMP and MROSMP of order m with respect to $\mathscr{S}$ for the special cases that $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,2m,H}$ and $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,2m-1,H}$ with relevance to Hermitian matrix fraction description.

Proposition 4.21. *Let $m \in \mathbb{N}_0$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,2m,H}$. Let $P_j \in \mathcal{P}_{p\times p,j,\mathbb{C}}$ be monic for $j \in \mathbb{Z}_{0,m}$. $(P_j)_{j=0}^m$ is an MLOSMP (resp. MROSMP) with respect to $\mathscr{S}$ if and only if $(P_j)_{j=0}^m$ is an orthogonalization of $(\mathcal{E}_j)_{j=0}^m$ with respect to $\psi_{\langle \mathbf{H}_m(\mathscr{S}),L\rangle}$ (resp. $\psi_{\langle \mathbf{H}_m(\mathscr{S}),R\rangle}$).*

Remark 4.22. Let $m \in \mathbb{N}_0$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,2m,H}$. Let $P_j \in \mathcal{P}_{p\times p,j,\mathbb{C}}$ be monic for $j \in \mathbb{Z}_{0,m}$. Then $(P_j)_{j=0}^m$ is an MLOSMP of order m with respect to $\mathscr{S}$ if and only if $(P_j^\vee)_{j=0}^m$ is an MROSMP of order m with respect to $\mathscr{S}$.

Proposition 4.23. *Let $m \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,2m,H}$. Then there exists an MLOSMP of order m with respect to $\mathscr{S}$ if and only if for $k \in \mathbb{Z}_{1,m}$,*

$$\mathrm{Ker}\mathbf{H}_{k-1}(\mathscr{S}) \subseteq \mathrm{Ker}\mathbf{H}^{\langle 1\rangle}_{k-1}(\mathscr{S}). \tag{4.14}$$

More specifically, let $\mathscr{S} \in \mathbb{C}^{p\times p}_{2m}$ be such that (4.14) *holds. Let $P_0(z) := I_p$ and let $P_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$ be monic for $k \in \mathbb{Z}_{1,m}$. Then $(P_k)_{k=0}^m$ is an MLOSMP of order m with respect to $\mathscr{S}$ if and only if for $k \in \mathbb{Z}_{1,m}$,*

$$\mathbf{H}^{\langle 1\rangle}_{k-1}(\mathscr{S}) = \mathbf{C}^{\langle 1\rangle}_{P_k}\mathbf{H}_{k-1}(\mathscr{S}). \tag{4.15}$$

Proof. Assume that (4.14) holds. Then for $k \in \mathbb{Z}_{0,m}$, there exists a monic matrix polynomial $P_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$ such that (4.15) holds. It follows that

$$Z_k^{[P_k]} \cdot \mathbf{H}^{\langle 0\rangle}_{k,k-1}(\mathscr{S}) = 0_{p\times kp}, \tag{4.16}$$

or, equivalently,

$$Z_k^{[P_k]}\begin{pmatrix}\psi_{\langle \mathbf{H}_k(\mathscr{S}),L\rangle}\left(\mathcal{E}_0,\mathcal{E}_j\right)\\ \vdots \\ \psi_{\langle \mathbf{H}_k(\mathscr{S}),L\rangle}\left(\mathcal{E}_k,\mathcal{E}_j\right)\end{pmatrix}=0_p,\ \forall j\in\mathbb{Z}_{0,k-1}. \tag{4.17}$$

By using the fact that $\psi_{\langle \mathbf{H}_k(\mathscr{S}),L\rangle}\in\Psi_L(\mathcal{P}_{p\times p,m,\mathbb{C}})$, we subsequently have

$$\psi_{\langle \mathbf{H}_k(\mathscr{S}),L\rangle}\left(P_k,\mathcal{E}_j\right)=0_p,\ \forall j\in\mathbb{Z}_{0,k-1}. \tag{4.18}$$

A combination of (4.17) and Remark 4.20 indicates that $(P_k)_{k=0}^m$ is an MLOSMP of order m with respect to $\mathscr{S}$.

Conversely, suppose that there exists an MLOSMP $(P_k)_{k=0}^m$ of order m with respect to $\mathscr{S}$. By Remark 4.20 we easily verify (4.18). Then due to the fact again that $\psi_{\langle \mathbf{H}_k(\mathscr{S}),L\rangle}\in\Psi_L(\mathcal{P}_{p\times p,m,\mathbb{C}})$, (4.17) is a direct consequence of (4.18). Then by (4.17) we can obtain successively (4.16), (4.15) and (4.14).

□

Analogously, we have

Proposition 4.24. *Let $m\in\mathbb{N}$ and let $\mathscr{S}\in\mathbb{C}^{p\times p}_{\infty,2m,H}$. Then there exists an MROSMP of order m with respect to $\mathscr{S}$ if and only if for $k\in\mathbb{Z}_{1,m}$,*

$$\operatorname{Coker}\mathbf{H}_{k-1}(\mathscr{S})\subseteq\operatorname{Coker}\mathbf{H}^{\langle 1\rangle}_{k-1}(\mathscr{S}). \tag{4.19}$$

More specifically, let $\mathscr{S}\in\mathbb{C}^{p\times p}_{\infty,2m,H}$ be such that (4.19) holds. Let $P_0(z):=I_p$ and let $P_k\in\mathcal{P}_{p\times p,k,\mathbb{C}}$ be monic for $k\in\mathbb{Z}_{1,m}$. Then $(P_k)_{k=0}^m$ is an MROSMP of order m with respect to $\mathscr{S}$ if and only if for $k\in\mathbb{Z}_{1,m}$,

$$\mathbf{H}^{\langle 1\rangle}_{k-1}(\mathscr{S})=\mathbf{H}_{k-1}(\mathscr{S})\mathbf{C}^{\langle 2\rangle}_{P_k}.$$

Proof. Use Proposition 4.23 and Remark 4.22. □

From Proposition 4.23 (resp. Proposition 4.24) we remark that the existence and formulation of MLOSMP (resp. MROSMP) of order m with respect to $\mathscr{S}$ depend uniquely on $\mathscr{S}_{\langle 2m-1\rangle}$. Hence Propositions 4.23 and 4.24 hold as well when the condition $\mathscr{S}\in\mathbb{C}^{p\times p}_{\infty,2m,H}$ is replaced by $\mathscr{S}\in\mathbb{C}^{p\times p}_{\infty,2m-1,H}$.

Remark 4.25. Let $m\in\mathbb{N}$ and let $\mathscr{S}\in\mathbb{C}^{p\times p}_{\infty,2m-1,H}$. From Proposition 4.23 we can see that there exists a unique MLOSMP (resp. MROSMP) of order m with respect to $\mathscr{S}$ if and only if $\mathscr{S}_{\langle 2m-2\rangle}\in\mathcal{H}^{>}_{p,2m-2}$.

Remark 4.26. Let $m\in\mathbb{N}$ and let $\mathscr{S}\in\mathbb{C}^{p\times p}_{\infty,2m-1,H}$. Then $(P_k)_{k=0}^m$ is a MLOSMP of order m with respect to $\mathscr{S}$ if and only if $(P_k^{\vee})_{k=0}^m$ is a MROSMP of order m with respect to $\mathscr{S}$.

In view of [17, Lemma E.3], one can see that

Remark 4.27. Let $m \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,2m-1,H}$. Let $(P_k)_{k=0}^m$ be a MLOSMP of order m with respect to $\mathscr{S}_{\langle 2m-1\rangle}$. Then for each $k \in \mathbb{Z}_{0,m}$, Q_k is left (resp. right) $\mathscr{S}$-associated with respective to P_k if and only if for each $k \in \mathbb{Z}_{0,m}$ and each $u \in \mathbb{C}$,

$$Q_k(u) = \psi_{\langle \mathbf{H}_m(\mathscr{S}),L\rangle}(P_k^{[u]}, \mathcal{E}_0),$$
$$(\text{resp. } Q_k(u) = \psi_{\langle \mathbf{H}_m(\mathscr{S}),R\rangle}(\mathcal{E}_0, P_k^{[u]}),)$$

where $P_k^{[u]}$ is the unique matrix polynomial in the set of $\mathcal{P}_{p\times p,\mathbb{C}}$ which fulfills

$$(z-u)P_k^{[u]}(z) = P_k(z) - P_k(u).$$

We conclude this chapter with the connection between NLFD (resp. NRFD) for Hermitian transfer function matrices and MLOSMP (resp. MROSMP).

Proposition 4.28. *Let $p, n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$. Let $G_k \in \mathscr{T}_{p,p}$ be Hermitian for each $k \in \mathbb{Z}_{1,[\frac{n}{2}]}$ and let $\mathscr{S}_{\langle 2k-1\rangle} \in \mathbb{C}^{p\times p}_{2k-1}$ be the $(2k-1)$-th SMP of G_k. Suppose that $P_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$ and $Q_k \in \mathcal{P}_{p\times p,\mathbb{C}}$ for each $k \in \mathbb{Z}_{0,[\frac{n}{2}]}$ and $P_0(z) := I_p$ and $Q_0(z) := 0_p$. Then for $k \in \mathbb{Z}_{1,[\frac{n}{2}]}$, the pair (Q_k, P_k) is a NLFD (resp. NRFD) for G_k if and only if both following conditions are satisfied:*

(i) *$(P_k)_{k=0}^{[\frac{n}{2}]}$ is a MLOSMP (resp. MROSMP) of order $[\frac{n}{2}]$ with respect to $\mathscr{S}$.*

(ii) *Q_k is left (resp. right) $\mathscr{S}$-associated with respect to P_k, $k \in \mathbb{Z}_{0,[\frac{n}{2}]}$.*

Proof. Apply Propositions 4.13–4.15, and Propositions 4.23 and 4.24. □

5 The Bezoutian of matrix polynomials and the inertia problem of matrix polynomials

The classical Bezoutian, introduced by Sylvester [48], plays an important role in the study of common divisors, common multiples and stability theory of scalar polynomials. While studying matrix polynomials, the matricial generalizations of Bezoutian arise naturally. For a more comprehensive investigation of Bezoutiants in the general case, the reader is referred to several early works by Anderson and Jury [4], by Bitmead et al. [10], and by Lerer and Tismenetsky [60].

In Section 5.1, we provide some preliminary results concerning the Anderson-Jury Bezoutian matrix for the most general case. In connection to some special transfer function matrices, we seek further important features of the Anderson-Jury Bezoutian matrix in Section 5.2. We show that, on the one hand, the Bezoutian matrix associated with a Hermitian transfer function matrix G is congruent to a block Hankel matrix generated by the SMP of G. On the other hand, the Bezoutian matrix associated with a unitary transfer function serves as a key instrument to calculate the inertia of matrix polynomials with respect to $\mathbb{R}$, as will be shown in Theorem 5.17.

5.1 Preliminaries

We begin this section with an introduction of the Anderson-Jury Bezoutian matrix.

Definition 5.1. Let $p \in \mathbb{N}$. Let L and $\widetilde{L} \in \mathcal{P}_{p\times p,\mathbb{C}}$. Moreover, let M, $\widetilde{M} \in \mathcal{P}_{p\times p,\mathbb{C}}$ be such that

$$\widetilde{M}(z)\widetilde{L}(z) = M(z)L(z), \quad z \in \mathbb{C}. \tag{5.1}$$

Then the *Anderson-Jury Bezoutian matrix* $\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L})$ *associated with the quadruple* $(\widetilde{M},\widetilde{L},M,L)$ is generated via the bilinear form associated with the quadruple $(\widetilde{M},\widetilde{L},M,L)$:

$$\begin{aligned}&(I_p, zI_p, \ldots, z^{m-1}I_p)\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L})(I_p, uI_p, \ldots, u^{l-1}I_p)^{\mathrm{T}}\\ =&\frac{1}{z-u}\left[\widetilde{M}(z)\widetilde{L}(u) - M(z)L(u)\right],\end{aligned} \tag{5.2}$$

where $l :=\max\{\deg L, \deg \widetilde{L}\}$ and $m :=\max\{\deg M, \deg \widetilde{M}\}$.

It is plain to see that the Anderson-Jury Bezoutian matrix $\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L})$ is skew-symmetric with respect to the quadruple $(\widetilde{M},\widetilde{L},M,L)$, i.e.

$$\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L}) = -\mathbf{B}_{M,\widetilde{M}}(\widetilde{L},L).$$

Our definition of the Anderson-Jury Bezoutian matrix $\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L})$ here seems slightly different from that in [59, pp. 389–390], in view of the fact that the associating matrix polynomials $\widetilde{M}$, $\widetilde{L}$, M and L are not necessarily confined to be regular. We admit that the main essential properties of this type of Bezoutian matrix are shown in the regular case. For some special reasons, however, more general matrix polynomials are inevitably involved in our investigation. Certainly we usually avoid the special case that $\widetilde{M} = M = 0_p$ and, in the event, $\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L}) = 0_{mp\times lp}$, which is totally insignificant. In the particular case that L and $\widetilde{L}$ commute, i.e., for each $z \in \mathbb{C}$,

$$L(z)\widetilde{L}(z) = \widetilde{L}(z)L(z),$$

the natural choice of $\widetilde{M}$ and M is that $\widetilde{M} = L$ and $M = \widetilde{L}$. For nontrivial choice of $\widetilde{M}$ and M in the generally non-commutative case, we refer the reader to the construction of the common multiples via spectral theory of matrix polynomials (see [38, Theorem 9.11] for the monic case and [41, Theorem 2.2] for the comonic case).

Let $n \in \mathbb{N}_0$ and let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be as in (2.1). Let also $\mathscr{S}^{[P]}_{\mathrm{I},n-1} := (A_j)_{j=0}^{n-1}$ and, for each $k \in \mathbb{N}$, $\mathscr{S}^{[P]}_{\mathrm{II},k} := (c_j)_{j=0}^{k}$, where

$$c_j := \begin{cases} A_{n-j}, & \text{if } j \in \mathbb{Z}_{0,n}, \\ 0_p, & \text{if } j \in \mathbb{Z}_{n+1,k}, \end{cases}$$

which are both uniquely determined by the representation of P. For simplicity we write $\mathscr{S}^{[P]}_{\mathrm{I}}$ and $\mathscr{S}^{[P]}_{\mathrm{II}}$ for $\mathscr{S}^{[P]}_{\mathrm{I},n-1}$ and $\mathscr{S}^{[P]}_{\mathrm{II},n-1}$, respectively. For a $p\times q$ matrix $A \in \mathbb{C}^{p\times q}$, we denote $A_{[j,k]}$ to be the $j\times k$ submatrix of A lying at the first j-th rows and the first k-th columns. The Anderson-Jury Bezoutian matrix $\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L})$ can be expressed in terms of the coefficients of the associated matrix polynomials L, $\widetilde{L}$, M and $\widetilde{M}$ as follows (see [60, (1.5), Section 1, p397]):

$$\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L}) = \begin{cases} \mathcal{S}^{\langle\mathrm{II}\rangle}_{m-1}(\mathscr{S}^{[\widetilde{M}]}_{\mathrm{I}}) \cdot \begin{pmatrix} \mathcal{S}^{\langle\mathrm{I}\rangle}_{l-1}(\mathscr{S}^{[\widetilde{L}]}_{\mathrm{II}}) \\ 0_{(m-l)p\times lp} \end{pmatrix} \\ \qquad -\mathcal{S}^{\langle\mathrm{II}\rangle}_{m-1}(\mathscr{S}^{[M]}_{\mathrm{I}}) \cdot \begin{pmatrix} \mathcal{S}^{\langle\mathrm{I}\rangle}_{l-1}(\mathscr{S}^{[L]}_{\mathrm{II}}) \\ 0_{(m-l)p\times lp} \end{pmatrix}, & m > l, \\ \mathcal{S}^{\langle\mathrm{II}\rangle}_{m-1}(\mathscr{S}^{[\widetilde{M}]}_{\mathrm{I}}) \cdot \left(\mathcal{S}^{\langle\mathrm{I}\rangle}_{l-1}(\mathscr{S}^{[\widetilde{L}]}_{\mathrm{II}})\right)_{[mp,lp]} \\ \qquad -\mathcal{S}^{\langle\mathrm{II}\rangle}_{m-1}(\mathscr{S}^{[M]}_{\mathrm{I}}) \cdot \left(\mathcal{S}^{\langle\mathrm{I}\rangle}_{m-1}(\mathscr{S}^{[L]}_{\mathrm{II}})\right)_{[mp,lp]}, & m \leq l. \end{cases} \tag{5.3}$$

Remark 5.2. [4, Remark 2.3] Let $p \in \mathbb{N}$. Let L, $\widetilde{L}$, M, $\widetilde{M} \in \mathcal{P}_{p\times p,\mathbb{C}}$ be such that (5.1) holds. Let $J \in \mathcal{P}_{p\times p,\mathbb{C}}$. Then

(i) For each $z \in \mathbb{C}$,

$$\begin{aligned}\widetilde{M}(z)\left(\widetilde{L} + J \cdot L\right)(z) &= \left(M + \widetilde{M} \cdot J\right)(z) \cdot L(z),\\ \left(\widetilde{M} + M \cdot J\right)(z) \cdot \widetilde{L}(z) &= M(z) \cdot \left(L + J \cdot \widetilde{L}\right)(z).\end{aligned}$$

(ii)

$$\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L}) = \mathbf{B}_{\widetilde{M},M+\widetilde{M}J}(L,\widetilde{L}+JL) = \mathbf{B}_{\widetilde{M}+MJ,M}(L+J\widetilde{L},\widetilde{L}).$$

Remark 5.3. Let $p \in \mathbb{N}$. Let L, $\widetilde{L}$, M, $\widetilde{M} \in \mathcal{P}_{p\times p,\mathbb{C}}$ be such that (5.1) holds. Then

$$\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L}) = \left(\mathbf{B}_{L^\vee,\widetilde{L}^\vee}(\widetilde{M}^\vee, M^\vee)\right)^*. \tag{5.4}$$

Proof. Let $l :=\max\{\deg\, L,\ \deg\, \widetilde{L}\}$ and $m :=\max\{\deg\, M,\ \deg\, \widetilde{M}\}$. By Definition 5.1, we have

$$\begin{aligned}&(I_p, zI_p, \ldots, z^{m-1}I_p)\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L})(I_p, uI_p, \ldots, u^{l-1}I_p)^{\mathrm{T}}\\ =&\frac{1}{z-u}[\widetilde{M}(z)\widetilde{L}(u) - M(z)L(u)]\\ =&\frac{1}{u-z}[L^\vee(\bar{u})M^\vee(\bar{z}) - \widetilde{L}^\vee(\bar{u})\widetilde{M}^\vee(\bar{z})]^*\\ =&\left((I_p, \bar{u}I_p, \ldots, \bar{u}^{l-1}I_p)\mathbf{B}_{L^\vee,\widetilde{L}^\vee}(\widetilde{M}^\vee, M^\vee)(I_p, \bar{z}I_p, \ldots, \bar{z}^{m-1}I_p)^{\mathrm{T}}\right)^*\\ =&(I_p, zI_p, \ldots, z^{m-1}I_p)\left(\mathbf{B}_{L^\vee,\widetilde{L}^\vee}(\widetilde{M}^\vee, M^\vee)\right)^*(I_p, uI_p, \ldots, u^{l-1}I_p)^{\mathrm{T}}.\end{aligned}$$

Hence (5.4) holds. □

The following result is an immediate consequence of Proposition 2.2 of [4].

Proposition 5.4. *Let $p \in \mathbb{N}$. Let L, $\widetilde{L}$, M, $\widetilde{M} \in \mathcal{P}_{p\times p,\mathbb{C}}$ be such that* (5.1) *holds. Moreover, let N, $\widetilde{N} \in \mathcal{P}_{p\times p,\mathbb{C}}$. Then*

(i) *For each $z \in \mathbb{C}$,*

$$(N\cdot\widetilde{M})(z)\cdot(\widetilde{L}\cdot\widetilde{N})(z) = (N\cdot M)(z)\cdot(L\cdot\widetilde{N})(z).$$

(ii)

$$\begin{aligned}&\begin{pmatrix}\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L}) & 0_{mp\times l_1p}\\ 0_{m_1p\times lp} & 0_{m_1p\times l_1p}\end{pmatrix}\\ =&\mathcal{S}^{\langle\mathrm{III}\rangle}_{m+m_1-1}(\mathscr{S}^{[N]}_{\mathrm{II},m+m_1-1})\mathbf{B}_{N\cdot\widetilde{M},N\cdot M}(L\cdot\widetilde{N},\widetilde{L}\cdot\widetilde{N})\mathcal{S}^{\langle\mathrm{I}\rangle}_{l+l_1-1}(\mathscr{S}^{[\widetilde{N}]}_{\mathrm{II},l+l_1-1}),\end{aligned}$$

where $m_1 :=\deg\, N$ and $l_1 :=\deg\, \widetilde{N}$.

Recall the definition of the inertia of a matrix: For $A \in \mathbb{C}^{p\times p}$, the inertia of A with respect to i$\mathbb{R}$ is defined by the triple

$$\mathrm{In}(A) := (\pi(A), \nu(A), \delta(A)),$$

where $\pi(A)$, $\nu(A)$, and $\delta(A)$ stand for the number of eigenvalues (counting algebraic multiplicities) of A with positive, negative, and zero real parts, respectively.

In the following we will introduce some extensions to the inertia of matrix polynomials, which are two of the most important notions in the thesis. We adopt the notations of [62], which also essentially coincide with the notations of [60] (see [62, Proposition 2.2]).

Definition 5.5. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$. Denote by $\gamma_+(F)$, $\gamma_-(F)$, $\gamma_0(F)$ the number of zeros of F (counting for multiplicities), lying in the open upper half plane $\mathbb{C}_U$, the open lower half plane $\mathbb{C}_L$ and on the real axis $\mathbb{R}$ (the multiplicity of infinity as a zero of F is defined to be equal to np), respectively. The triple

$$\gamma(F) := (\gamma_+(F), \gamma_-(F), \gamma_0(F))$$

is called *the inertia of F with respect to* $\mathbb{R}$. Analogously, the triple

$$\gamma'(F) := (\gamma'_+(F), \gamma'_-(F), \gamma'_0(F))$$

is called *the inertia of F with respect to* i$\mathbb{R}$, replacing open upper half plane by the open right half plane $\mathbb{C}_+$, the open lower half plane by the open left half plane $\mathbb{C}_-$, and the real axis $\mathbb{R}$ (including infinity) by the imaginary axis i$\mathbb{R}$ (including infinity), respectively.

By applying Proposition 5.4, we can easily have

Proposition 5.6. *Let p, $n \in \mathbb{N}$. Let L, $\widetilde{L}$, M, $\widetilde{M} \in \mathcal{P}_{p\times p,m,\mathbb{C}}$ be such that* (5.1) *holds. Moreover, let $N \in \mathcal{P}_{p\times p,\mathbb{C}}$. Then*

$$\mathrm{In}\left(\mathbf{B}_{N\cdot\widetilde{M},N\cdot M}(L\cdot N^{\vee}, \widetilde{L}\cdot N^{\vee})\right) = \mathrm{In}\left(\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L})\right) + (0,0,m_1 p),$$

where $m_1 :=$deg N.

In the following we will show the behaviour of Anderson-Jury Bezoutian matrices under certain linear combinations of matrix polynomials.

Proposition 5.7. *Let L and $\widetilde{L} \in \mathcal{P}_{p\times p,\mathbb{C}}$. Let M and $\widetilde{M} \in \mathcal{P}_{p\times p,\mathbb{C}}$ be such that* (5.1) *holds. Let a, b, c, $d \in \mathbb{C}$ be such that $ad - bc \neq 0$. Let, for $z \in \mathbb{C}$,*

$$\widetilde{M}^{\diamond}(z) := a\widetilde{M}(z) + cM(z), \quad M^{\diamond}(z) := b\widetilde{M}(z) + dM(z),$$
$$L^{\diamond}(z) := aL(z) + c\widetilde{L}(z), \quad \widetilde{L}^{\diamond}(z) := bL(z) + d\widetilde{L}(z).$$

(i) *For $z \in \mathbb{C}$,*

$$\widetilde{M}^{\diamond}(z)\widetilde{L}^{\diamond}(z) = M^{\diamond}(z)L^{\diamond}(z).$$

(ii)
$$\begin{pmatrix} \mathbf{B}_{\widetilde{M}^\diamond,M^\diamond}(L^\diamond,\widetilde{L}^\diamond) & 0_{m_1p\times(l-l_1)p} \\ 0_{(m-m_1)p\times l_1p} & 0_{(m-m_1)p\times(l-l_1)p} \end{pmatrix} = (ad-bc)\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L}),$$

where l :=max{deg L, deg $\widetilde{L}$}, l_1 :=max{deg $L^\diamond$, deg $\widetilde{L}^\diamond$}, m :=max{deg M, deg $\widetilde{M}$} *and* m_1 :=max{deg $M^\diamond$, deg $\widetilde{M}^\diamond$}.

Proof. By straightforward calculation we have, for z, $u \in \mathbb{C}$,

$$\begin{aligned} &\widetilde{M}^\diamond(z)\widetilde{L}^\diamond(u) - M^\diamond(z)L^\diamond(u) \\ =&(ad-bc)(\widetilde{M}(z)\widetilde{L}(u) - M(z)L(u)). \end{aligned} \tag{5.5}$$

Substituting z for u in the formula (5.5) gives (i).

By plugging (5.5) into (5.2), we obtain that

$$\begin{aligned} &(I_p, zI_p, \ldots, z^{m-1}I_p)\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L})(I_p, uI_p, \ldots, u^{l-1}I_p)^{\mathrm{T}} \\ =&\frac{\widetilde{M}^\diamond(z)\widetilde{L}^\diamond(u) - M^\diamond(z)L^\diamond(u)}{(ad-bc)(z-u)} \\ =&(ad-bc)^{-1}(I_p, zI_p, \ldots, z^{m_1-1}I_p)\mathbf{B}_{\widetilde{M}^\diamond,M^\diamond}(L^\diamond,\widetilde{L}^\diamond)(I_p, uI_p, \ldots, u^{l_1-1}I_p)^{\mathrm{T}} \\ =&(I_p, \ldots, z^{m-1}I_p)(ad-bc)^{-1}\begin{pmatrix} \mathbf{B}_{\widetilde{M}^\diamond,M^\diamond}(L^\diamond,\widetilde{L}^\diamond) & 0_{m_1p\times(l-l_1)p} \\ 0_{(m-m_1)p\times l_1p} & 0_{(m-m_1)p\times(l-l_1)p} \end{pmatrix}\begin{pmatrix} I_p \\ \vdots \\ u^{l-1}I_p \end{pmatrix}, \end{aligned}$$

which implies (ii). □

For a given pair of matrix polynomials L and $\widetilde{L}$, the Anderson-Jury Bezoutian matrix $\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L})$ depends on the matrix polynomials M, $\widetilde{M}$ chosen to verify (5.1). However it should be pointed out that, in the regular case, the dimension of the kernel of $\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L})$ is independent of the chosen $\widetilde{M}$ and M. More precisely, it coincides with the number of all the zeros (counting algebraic multiplicities) of a g.r.c.d of L and $\widetilde{L}$:

Proposition 5.8. *[60, Theorem 0.2] Let* $p \in \mathbb{N}$. *Let* L *and* $\widetilde{L} \in \mathcal{P}_{p\times p,\mathbb{C}}$ *be regular. Moreover, let* M, $\widetilde{M} \in \mathcal{P}_{p\times p,\mathbb{C}}$ *be regular such that* (5.1) *holds. Then*

$$\dim\mathrm{Ker}\mathbf{B}_{\widetilde{M},M}(L,\widetilde{L}) = \deg(\det L_0(z)),$$

where L_0 *is a g.r.c.d of* L *and* $\widetilde{L}$.

As a consequence, we give a dual result of Proposition 5.8.

Proposition 5.9. *Let* $p \in \mathbb{N}$. *Let* L *and* $\widetilde{L} \in \mathcal{P}_{p\times p,\mathbb{C}}$ *be regular. Moreover, let* M, $\widetilde{M} \in \mathcal{P}_{p\times p,\mathbb{C}}$ *be regular such that*

$$L(z)M(z) = \widetilde{L}(z)\widetilde{M}(z), \quad z \in \mathbb{C}. \tag{5.6}$$

Then

$$\dim\mathrm{Ker}\mathbf{B}_{L,\widetilde{L}}(\widetilde{M}, M) = \deg(\det L_0(z)) + (l-m)p, \tag{5.7}$$

where L_0 is a g.l.c.d of L and $\widetilde{L}$, $l :=$max$\{$deg L, deg $\widetilde{L}\}$ and $m :=$max$\{$deg M, deg $\widetilde{M}\}$.

Proof. Substituting $\widetilde{M}^\vee$, $M^\vee$, $L^\vee$ and $\widetilde{L}^\vee$ for $\widetilde{M}$, M, L and $\widetilde{L}$ in Proposition 5.8, we have

$$\dim\mathrm{Ker}\mathbf{B}_{\widetilde{M}^\vee, M^\vee}(L^\vee, \widetilde{L}^\vee) = \deg(\det\widetilde{L}_0(z)), \tag{5.8}$$

where $\widetilde{L}_0$ is a g.r.c.d of $L^\vee$ and $\widetilde{L}^\vee$.

Suppose that $L_0(z) := \widetilde{L}_0^\vee(z)$ for each $z \in \mathbb{C}$. In view of Remark 5.3, we turn (5.8) into

$$\begin{aligned}\dim\mathrm{Ker}\mathbf{B}_{L,\widetilde{L}}(\widetilde{M}, M) - (l-m)p =& \dim\mathrm{Ker}\left(\mathbf{B}_{L,\widetilde{L}}(\widetilde{M}, M)\right)^* \\ =& \dim\mathrm{Ker}\mathbf{B}_{\widetilde{M}^\vee, M^\vee}(L^\vee, \widetilde{L}^\vee) \\ =& \deg(\det\widetilde{L}_0(z)) = \deg(\det L_0(z)).\end{aligned} \tag{5.9}$$

Since (2.18) implies that L_0 is a g.l.c.d of $L^\vee$ and $\widetilde{L}^\vee$, the equality (5.7) is an immediate consequence of (5.9).

□

By applying Propositions 5.8 and 5.9, we obtain criteria for coprimeness of matrix polynomials, respectively.

Proposition 5.10. *Let $p \in \mathbb{N}$. Let L and $\widetilde{L} \in \mathcal{P}_{p\times p,\mathbb{C}}$ be regular. Moreover, let M, $\widetilde{M} \in \mathcal{P}_{p\times p,\mathbb{C}}$ be such that* (5.1) *holds. Then L and $\widetilde{L}$ are right coprime if and only if*

$$\dim\mathrm{Ker}\mathbf{B}_{\widetilde{M},M}(L, \widetilde{L}) = 0.$$

Proposition 5.11. *Let $p \in \mathbb{N}$. Let L and $\widetilde{L} \in \mathcal{P}_{p\times p,\mathbb{C}}$ be regular. Moreover, let M, $\widetilde{M} \in \mathcal{P}_{p\times p,\mathbb{C}}$ be such that* (5.6) *holds. Then L and $\widetilde{L}$ are left coprime if and only if*

$$\dim\mathrm{Ker}\mathbf{B}_{L,\widetilde{L}}(\widetilde{M}, M) = (l-m)p,$$

where $l :=$max$\{$deg L, deg $\widetilde{L}\}$ and $m :=$max$\{$deg M, deg $\widetilde{M}\}$.

5.2 The Anderson-Jury Bezoutian matrices in connection to special transfer function matrices

In this section, we will investigate the Anderson-Jury Bezoutian matrix associated with some special transfer function matrices and derive further important features.

Starting from a Hermitian transfer function G and its NRFD (N, D), we draw our attention to the particular Anderson-Jury Bezoutian matrix $\mathbf{B}_{D^\vee,N^\vee}(D, N)$. We will now focus on proving that this particular Anderson-Jury Bezoutian matrix is congruent to a block Hankel matrix generated by the SMP of G.

Proposition 5.12. *([4, Lemma 2.3]) Let $p \in \mathbb{N}$. Let $G \in \mathcal{T}^{\diamond}_{p,p}$ be Hermitian and let (N, D) be an NRFD for G. Let $\mathscr{S}$ be the SMP of G. Assume that $m :=\deg D$ and D is written as follows:*

$$D(z) = \sum_{k=0}^{m} D_{m-k} z^k,$$

where $D_k \in \mathbb{C}^{p\times p}$, $k \in \mathbb{Z}_{0,m}$. Then

$$\mathbf{B}_{D^\vee,N^\vee}(D,N) = \begin{pmatrix} D^*_{m-1} & \cdots & D^*_0 \\ \vdots & \iddots & \\ D^*_0 & & \end{pmatrix} \mathbf{H}_{m-1}(\mathscr{S}) \begin{pmatrix} D_{m-1} & \cdots & D_0 \\ \vdots & \iddots & \\ D_0 & & \end{pmatrix}.$$

Moreover,

$$\mathrm{In}(\mathbf{B}_{D^\vee,N^\vee}(D,N)) = \mathrm{In}(\mathbf{H}_{m-1}(\mathscr{S})).$$

Substituting $G^\vee$, $N^\vee$ and $D^\vee$ for G, N and D in Proposition 5.12, respectively, we obtain the dual conclusion.

Proposition 5.13. *Let $p \in \mathbb{N}$. Let $G \in \mathcal{T}^{\diamond}_{p,p}$ be Hermitian and let (N, D) be a NLFD for G. Let $\mathscr{S}$ be the SMP of G. Assume that $m :=\deg D$ and D is written as follows:*

$$D(z) = \sum_{k=0}^{m} D_{m-k} z^k,$$

where $D_k \in \mathbb{C}^{p\times p}$ for $k \in \mathbb{Z}_{0,m}$. Then

$$\mathbf{B}_{D,N}(D^\vee,N^\vee) = \begin{pmatrix} D_{m-1} & \cdots & D_0 \\ \vdots & \iddots & \\ D_0 & & \end{pmatrix} \mathbf{H}_{m-1}(\mathscr{S}) \begin{pmatrix} D^*_{m-1} & \cdots & D^*_0 \\ \vdots & \iddots & \\ D^*_0 & & \end{pmatrix}. \tag{5.10}$$

Moreover,

$$\mathrm{In}(\mathbf{B}_{D,N}(D^\vee,N^\vee)) = \mathrm{In}(\mathbf{H}_{m-1}(\mathscr{S})). \tag{5.11}$$

Proposition 5.14. *Let $p \in \mathbb{N}$. Let $G \in \mathcal{T}^{\diamond}_{p,p}$ be Hermitian and let (N, D) be a NRFD for G. Let $\mathscr{S}$ be the SMP of G.*

$$\begin{aligned} \mathrm{dimKer}\mathbf{H}_{m-1}(\mathscr{S}) =& \mathrm{dimKer}\mathbf{B}_{D,N}(D^\vee,N^\vee) \\ =& \mathrm{dimKer}\mathbf{B}_{D^\vee,N^\vee}(D,N) \\ =& \deg(\det N_0(z)), \end{aligned} \tag{5.12}$$

where N_0 is a g.r.c.d of D and N.

Proof. Suppose that two monic matrix polynomials L, $L_1 \in \mathcal{P}_{p\times p,\mathbb{C}}$ are defined via

$$L(z) := D(z) + N(z), \quad L_1(z) := D(z) - N(z).$$

Then by using Proposition 5.7 we obtain that

$$L(z)L_1^{\vee}(z) = L_1(z)L^{\vee}(z)$$

and

$$\mathrm{dimKer}\mathbf{B}_{L,L_1}(L^{\vee}, L_1^{\vee}) = \mathrm{dimKer}\mathbf{B}_{D,N}(D^{\vee}, N^{\vee}). \tag{5.13}$$

Proposition 2.17 gives that N_0 is a g.r.c.d of L and L_1. Since the four matrix polynomials L, L_1, $L^{\vee}$ and $L_1^{\vee}$ are monic and then regular, we have

$$\begin{aligned}\mathrm{dimKer}\mathbf{H}_{m-1}(\mathscr{S}) =& \mathrm{dimKer}\mathbf{B}_{D,N}(D^{\vee}, N^{\vee})\\ =& \mathrm{dimKer}\mathbf{B}_{L,L_1}(L^{\vee}, L_1^{\vee})\\ =& \deg(\det N_0(z)).\end{aligned} \tag{5.14}$$

where the 1st equality follows from Proposition 5.13, the 2nd equality is (5.13) and the 3rd equality is due to Proposition 5.8.

Since $N_0^{\vee}$ is a g.l.c.d of $D^{\vee}$ and $N^{\vee}$ due to Proposition 2.18, applying Proposition 5.9 gives that

$$\deg(\det N_0(z)) = \deg(\det N_0^{\vee}(z)) = \mathrm{dimKer}\mathbf{B}_{D^{\vee},N^{\vee}}(D, N). \tag{5.15}$$

Then (5.12) is a combination of (5.14) and (5.15). □

Applying Proposition 4.15 and replacing G, N and D by, respectively, $G^{\vee}$, $N^{\vee}$ and $D^{\vee}$ in Proposition 5.14 yield that

Proposition 5.15. *Let $p \in \mathbb{N}$. Let $G \in \mathcal{T}_{p,p}^{\diamond}$ be Hermitian and let (N, D) be a NLFD for G. Let $\mathscr{S}$ be the SMP of G.*

$$\begin{aligned}\mathrm{dimKer}\mathbf{H}_{m-1}(\mathscr{S}) =& \mathrm{dimKer}\mathbf{B}_{D,N}(D^{\vee}, N^{\vee})\\ =& \mathrm{dimKer}\mathbf{B}_{D^{\vee},N^{\vee}}(D, N)\\ =& \deg(\det N_0(z)),\end{aligned}$$

where N_0 is a g.l.c.d of D and N.

According to Propositions 5.14 and 5.15, one can check whether the SRMP or SLMP of a Hermitian transfer function matrix G is reducible by testing a block Hankel matrix generated by the SMP of G (see Proposition 5.16 below).

Proposition 5.16. *Let $p \in \mathbb{N}$. Let G be Hermitian and let (N, D) be an NRFD (resp. NLFD) for G. Moreover, let $\mathscr{S}$ be the SMP of G and let $m :=\deg D$. Then (N, D) is irreducible if and only if $\mathbf{H}_{m-1}(\mathscr{S})$ is nonsingular.*

Our next investigation concerns the role of the Bezoutian matrix in solving an inertia problem for matrix polynomials.

Suppose that $F \in \mathcal{P}_{p\times p,\mathbb{C}}$. How to count $\gamma(F)$ is a fundamental problem in the theory of stability. In the scalar case, the relation between $\gamma(F)$ and the inertia of an appropriate Hermitian matrix with respect to $\mathrm{i}\mathbb{R}$ is revealed by a celebrated result due to Fujiwara (see, e.g., [36], [59, pp. 466–467]). What concerns the matrix case the reader is referred to [23, 60, 62]. In the following we point out a particular refinement of the classical Hermite-Fujiwara theorem by Lerer and Tismenetsky [60], which is based on the spectral theory of matrix polynomials (see a comprehensive study in the monograph [38]). This refinement demonstrates that for a given regular matrix polynomial F, the inertia $\gamma(F)$ is described via a Anderson-Jury Bezoutian matrix in connection to a unitary transfer function matrix.

Theorem 5.17. *(see, e.g., [60, Theorem 2.1]) Let $p \in \mathbb{N}$ and let $F \in \mathcal{P}_{p\times p,\mathbb{C}}$ be regular. Let also $G \in \mathscr{T}_{p\times p}$ be unitary with an RFD (F, F_1), where $F_1 \in \mathcal{P}_{p\times p,\mathbb{C}}$ is regular. Then*

$$\begin{aligned}
\gamma_+(F) &= \pi(-\mathrm{i}\mathbf{B}_{F_1^\vee,F^\vee}(F, F_1)) + \gamma_+(F_0),\\
\gamma_-(F) &= \nu(-\mathrm{i}\mathbf{B}_{F_1^\vee,F^\vee}(F, F_1)) + \gamma_-(F_0),\\
\gamma_0(F) &= \delta(-\mathrm{i}\mathbf{B}_{F_1^\vee,F^\vee}(F, F_1)) - \gamma_+(F_0) - \gamma_-(F_0),
\end{aligned}$$

where F_0 is a g.r.c.d of F and F_1.

The aim of Theorem 5.17 is a description of the inertia of F in terms of the inertia of $-\mathrm{i}\mathbf{B}_{F_1^\vee,F^\vee}(F, F_1)$. In general, the inertia of $-\mathrm{i}\mathbf{B}_{F_1^\vee,F^\vee}(F, F_1)$ depends on the choice of F_1. The trivial choice $F_1 = F$ implies that $\mathbf{B}_{F_1^\vee,F^\vee}(F, F_1) = 0$ and reveals no information on the location of the spectrum of F. In the scalar case the obvious choice $F_1 = F^\vee$ leads to the classical Bezoutian which occurs in the Hermite-Fujiwara theorem.

Substituting $F^\vee$ and $F_1^\vee$ for F and F_1 in Theorem 5.17, respectively, we have

Corollary 5.18. *Let $p \in \mathbb{N}$ and let $F \in \mathcal{P}_{p\times p,\mathbb{C}}$ be regular. Moreover, let $G \in \mathscr{T}_{p\times p}$ be unitary with a LFD (F, F_1), where $F_1 \in \mathcal{P}_{p\times p,\mathbb{C}}$ is regular. Then*

$$\begin{aligned}
\gamma_+(F) &= \pi(-\mathrm{i}\mathbf{B}_{F_1,F}(F^\vee, F_1^\vee)) + \gamma_+(F_0),\\
\gamma_-(F) &= \nu(-\mathrm{i}\mathbf{B}_{F_1,F}(F^\vee, F_1^\vee)) + \gamma_-(F_0),\\
\gamma_0(F) &= \delta(-\mathrm{i}\mathbf{B}_{F_1,F}(F^\vee, F_1^\vee)) - \gamma_+(F_0) - \gamma_-(F_0),
\end{aligned}$$

where F_0 is a g.l.c.d of F and F_1.

6 Para-Hermitian strictly proper transfer function matrices and their related monic Hurwitz matrix polynomials

In this chapter, we will relate matrix fraction descriptions for para-Hermitian strictly proper transfer function matrices to Hurwitz matrix polynomials. Recall that for $n \in \mathbb{N}_0$ and $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$, F is called a *Hurwitz matrix polynomial* if $\sigma(F) \subseteq \mathbb{C}_-$, or equivalently, $\gamma'_+(F) = \gamma'_0(F) = 0$, $\gamma'_-(F) = np$.

For the scalar case $p = 1$, characterization of such a polynomial is used to indicate stability or instability of dynamical motions (see Routh [68]). They gave several general linear methods to check whether a polynomial is a polynomial of this type or not. What concerns the generalization of such polynomials to matrix polynomials, we refer to Choque Rivero [17]. Furthermore, one can look at entire functions and impose restrictions on their zero locations similar to those on Hurwitz polynomials, see e.g. Grommer [42], Chebotarev/Meiman [11], Krein [56], Levin [63], Katsnelson [53].

This chapter mainly concerns the following question: Given a transfer function matrix G and its NRFD (resp. NLFD) (N, D), how can we use the SMP of G to determine whether a linear combination $aN + bD$ $(\operatorname{Im}(a\bar{b}) \neq 0)$ between matrix polynomials N and D over $\mathbb{C}$ is a Hurwitz matrix polynomial or not?

To give an explicit answer to this question, we start with some connections between several types of transfer function matrices and the inertia of certain matrix polynomials constructed by their matrix fraction descriptions, which is based on the matricial refinement of Hermitian-Fujiwara theorem (Theorem 5.17) shown in Chapter 5. We also highlight that some of the connections play a pivotal role in the analysis of Chapter 8.

Proposition 6.1. *Let $p \in \mathbb{N}$. Let $G \in \mathcal{T}^{\diamond}_{p\times p}$ be Hermitian and let (N, D) be an NRFD for G. Let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty}$ be the SMP of G.*

(i) *For each pair $(a,\, b) \in \mathbb{C}\times\mathbb{C}$ such that $\operatorname{Im}(a\bar{b}) > 0$,*

$$\begin{aligned}
\gamma_+(aN + bD) &= \nu(\mathbf{H}_{m-1}(\mathscr{S})) + \gamma_+(E),\\
\gamma_-(aN + bD) &= \pi(\mathbf{H}_{m-1}(\mathscr{S})) + \gamma_-(E),\\
\gamma_0(aN + bD) &= \delta(\mathbf{H}_{m-1}(\mathscr{S})) - \gamma_+(E) - \gamma_-(E),
\end{aligned}$$

where E is a g.r.c.d of N and D and $m :=$ deg D. Moreover, suppose that

(N, D) is irreducible. Then

$$\begin{aligned}\gamma_+(aN+bD) &= \nu(\mathbf{H}_{m-1}(\mathscr{S})),\\ \gamma_-(aN+bD) &= \pi(\mathbf{H}_{m-1}(\mathscr{S})),\\ \gamma_0(aN+bD) &= \delta(\mathbf{H}_{m-1}(\mathscr{S})).\end{aligned}$$

(ii) *For each pair $(a,\, b) \in \mathbb{C} \times \mathbb{C}$ such that* $\mathrm{Im}(a\bar{b}) < 0$,

$$\begin{aligned}\gamma_+(aN+bD) &= \pi(\mathbf{H}_{m-1}(\mathscr{S})) + \gamma_+(E),\\ \gamma_-(aN+bD) &= \nu(\mathbf{H}_{m-1}(\mathscr{S})) + \gamma_-(E),\\ \gamma_0(aN+bD) &= \delta(\mathbf{H}_{m-1}(\mathscr{S})) - \gamma_+(E) - \gamma_-(E),\end{aligned}$$

where E is a g.r.c.d of N and D and $m :=$ deg D. Moreover, suppose that (N, D) is irreducible. Then

$$\begin{aligned}\gamma_+(aN+bD) &= \nu(\mathbf{H}_{m-1}(\mathscr{S})),\\ \gamma_-(aN+bD) &= \pi(\mathbf{H}_{m-1}(\mathscr{S})),\\ \gamma_0(aN+bD) &= \delta(\mathbf{H}_{m-1}(\mathscr{S})).\end{aligned}$$

Proof. Suppose that $(a,\, b) \in \mathbb{C} \times \mathbb{C}$ is a pair such that $\mathrm{Im}(a\bar{b}) \neq 0$. Let, for $z \in \mathbb{C}$,

$$\begin{aligned}L(z) &:= N(z) + a^{-1}b \cdot D(z),\\ L_1(z) &:= N(z) + \overline{a^{-1}b} \cdot D(z).\end{aligned}$$

Then we have

$$\begin{aligned}L^\vee(z) &= N^\vee(z) + \overline{a^{-1}b} \cdot D^\vee(z),\\ L_1^\vee(z) &= N^\vee(z) + a^{-1}b \cdot D^\vee(z).\end{aligned}$$

Obviously L and L_1 are regular. Applying Proposition 5.7 we have

$$\mathbf{B}_{L_1,L}(L^\vee, L_1^\vee) = 2\mathrm{i}\mathrm{Im}(a^{-1}b)\mathbf{B}_{N,D}(N^\vee, D^\vee).$$

A combination of the fact that

$$\begin{pmatrix} L(z)\\ L_1(z)\end{pmatrix} = \begin{pmatrix} I_p & a^{-1}b \cdot I_p\\ I_p & \overline{a^{-1}b} \cdot I_p\end{pmatrix}\begin{pmatrix} N(z)\\ D(z)\end{pmatrix}, \quad z \in \mathbb{C}.$$

and Proposition 2.19 shows that E is a g.r.c.d of L and L_1.

It follows that

$$\begin{aligned}\gamma_+(aN+bD) = \gamma_+(L) &= \pi(-\mathrm{i}\mathbf{B}_{L_1^\vee,L^\vee}(L, L_1)) + \gamma_+(E)\\ &= \pi(\mathrm{Im}(a^{-1}b) \cdot \mathbf{B}_{N^\vee,D^\vee}(N, D)) + \gamma_+(E)\\ &= \pi(-\mathrm{Im}(a\bar{b}) \cdot \mathbf{H}_{m-1}(\mathscr{S})) + \gamma_+(E),\end{aligned}$$

where the 2nd equality is due to Theorem 5.17, and the last equality is due to Proposition 5.12 and the fact that $(a\bar{a}) \cdot \mathrm{Im}(a^{-1}b) = -\mathrm{Im}(a\bar{b})$.

Analogously, we have

$$\begin{aligned}\gamma_-(aN + bD) &= \nu(-\mathrm{Im}(a\bar{b}) \cdot \mathbf{H}_{m-1}(\mathscr{S})) + \gamma_-(E),\\ \gamma_0(aN + bD) &= \delta(-\mathrm{Im}(a\bar{b}) \cdot \mathbf{H}_{m-1}(\mathscr{S})) - \gamma_+(E) - \gamma_-(E).\end{aligned}$$

□

Proposition 6.2. *Let $p \in \mathbb{N}$. Let $G \in \mathcal{T}^{\diamond}_{p\times p}$ be para-Hermitian and let (N, D) be an NRFD for G.*

(i) *For each pair $(a,\ b) \in \mathbb{C} \times \mathbb{C}$ such that $\mathrm{Im}(a\bar{b}) > 0$,*

$$\gamma'_+(aN + bD) = \nu(\mathbf{H}_{m-1}(\mathscr{S})) + \gamma'_+(E), \tag{6.1}$$
$$\gamma'_-(aN + bD) = \pi(\mathbf{H}_{m-1}(\mathscr{S})) + \gamma'_-(E), \tag{6.2}$$
$$\gamma'_0(aN + bD) = \delta(\mathbf{H}_{m-1}(\mathscr{S})) - \gamma'_+(E) - \gamma'_-(E), \tag{6.3}$$

where $\mathscr{S} := (\mathrm{i}^{k+1}s_k)_{k=0}^{\infty}$, $(s_k)_{k=0}^{\infty}$ is the SMP of G, E is a g.r.c.d of N and D and $m :=$ deg D. Moreover, if (N, D) is irreducible, then

$$\gamma'_+(aN + bD) = \nu(\mathbf{H}_{m-1}(\mathscr{S})), \tag{6.4}$$
$$\gamma'_-(aN + bD) = \pi(\mathbf{H}_{m-1}(\mathscr{S})), \tag{6.5}$$
$$\gamma'_0(aN + bD) = \delta(\mathbf{H}_{m-1}(\mathscr{S})). \tag{6.6}$$

(ii) *For each pair $(a,\ b) \in \mathbb{C} \times \mathbb{C}$ such that $\mathrm{Im}(a\bar{b}) < 0$, (6.1)–(6.3) hold, where $\mathscr{S} := ((-\mathrm{i})^{k+1}s_k)_{k=0}^{\infty}$, $(s_k)_{k=0}^{\infty}$ is the SMP of G and E is a g.r.c.d of N and D. Moreover, if (N, D) is irreducible, then (6.4)–(6.6) hold.*

Proof. The proof for (i): Let $\widetilde{G}(z) := G(-\mathrm{i}z)$, $\widetilde{N}(z) := \mathrm{i}^m N(-\mathrm{i}z)$ and $\widetilde{D}(z) := \mathrm{i}^m D(-\mathrm{i}z)$, where $m :=$deg D. Then obviously $\widetilde{G} \in \mathcal{T}^{\diamond}_{p\times p}$ is Hermitian, $\mathscr{S}$ is the SMP of $\widetilde{G}$ and $(\widetilde{N}, \widetilde{D})$ is an NRFD for $\widetilde{G}$. For each pair $(a,\ b) \in \mathbb{C} \times \mathbb{C}$ such that $\mathrm{Im}(a\bar{b}) > 0$, we have

$$\gamma'_-(aN + bD) = \gamma_-(a\widetilde{N} + b\widetilde{D}), \tag{6.7}$$
$$\gamma'_+(aN + bD) = \gamma_+(a\widetilde{N} + b\widetilde{D}), \tag{6.8}$$
$$\gamma'_0(aN + bD) = \gamma_0(a\widetilde{N} + b\widetilde{D}). \tag{6.9}$$

By using (6.7)–(6.9) and substituting $\widetilde{G}$, $\widetilde{N}$ and $\widetilde{D}$ for G, N and D in Proposition 6.1, respectively, we complete the proof for (i).

The proof for (ii): Let $\widetilde{G}(z) := G(\mathrm{i}z)$, $\widetilde{N}(z) := (-\mathrm{i})^m N(\mathrm{i}z)$ and $\widetilde{D}(z) := (-\mathrm{i})^m D(\mathrm{i}z)$, where $m :=$deg D. Then obviously $\widetilde{G} \in \mathcal{T}^{\diamond}_{p\times p}$ is Hermitian, $\mathscr{S}$ is the SMP of $\widetilde{G}$ and

$(\widetilde{N}, \widetilde{D})$ is an NRFD for $\widetilde{G}$. For each pair $(a, b) \in \mathbb{C} \times \mathbb{C}$ such that $\operatorname{Im}(a\bar{b}) < 0$, we have

$$\gamma'_-(aN + bD) = \gamma_+(a\widetilde{N} + b\widetilde{D}), \tag{6.10}$$

$$\gamma'_+(aN + bD) = \gamma_-(a\widetilde{N} + b\widetilde{D}), \tag{6.11}$$

$$\gamma'_0(aN + bD) = \gamma_0(a\widetilde{N} + b\widetilde{D}). \tag{6.12}$$

By using (6.10)–(6.12) and substituting $\widetilde{G}$, $\widetilde{N}$ and $\widetilde{D}$ for, respectively, G, N and D in Proposition 6.1, we complete the proof for (ii). □

Now we can seek the connection between para-Hermitian strictly proper transfer function matrices and monic Hurwitz matrix polynomials.

Proposition 6.3. *Let $p \in \mathbb{N}$. Let $G \in \mathcal{T}^{\diamond}_{p\times p}$ be para-Hermitian and let (N, D) be an NRFD for G. Let $\mathscr{S} := (\mathrm{i}^{k+1} s_k)_{k=0}^{\infty}$, where $(s_k)_{k=0}^{\infty}$ is the SMP of G. Then the following statements are equivalent:*

(i) *There exists a pair a_0, $b_0 \in \mathbb{C}$ satisfying $\operatorname{Im}(a_0\bar{b}_0) > 0$ such that $a_0N + b_0D$ is a Hurwitz matrix polynomial.*

(ii) *$\mathscr{S}_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$, where $m := \deg D$.*

(iii) *For each pair a, $b \in \mathbb{C}$ satisfying $\operatorname{Im}(a\bar{b}) > 0$, $aN + bD$ is a Hurwitz matrix polynomial.*

Proof. The implication "(i)$\Longrightarrow$ (ii)": Suppose that there exists a pair a_0, $b_0 \in \mathbb{C}$ such that $\operatorname{Im}(a_0\bar{b}_0) > 0$ and $a_0N + b_0D$ is a Hurwitz matrix polynomial. Then

$$\gamma'_+(aN + bD) = \gamma'_0(aN + bD) = 0.$$

It follows from Proposition 6.2 that

$$\nu(\mathbf{H}_n(\mathscr{S})) = \delta(\mathbf{H}_n(\mathscr{S})) = 0,$$

or equivalently, $\mathscr{S}_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$.

The implication "(ii)$\Longrightarrow$ (iii)" is an immediate consequence of Proposition 6.2 and "(iii)$\Longrightarrow$ (i)" is obvious. □

Proposition 6.4. *Let $p \in \mathbb{N}$. Let $G \in \mathcal{T}^{\diamond}_{p\times p}$ be para-Hermitian and let (N, D) be an NRFD for G. Let $\mathscr{S} := ((-\mathrm{i})^{k+1} s_k)_{k=0}^{\infty}$, where $(s_k)_{k=0}^{\infty}$ is the SMP of G. Then the following statements are equivalent:*

(i) *There exists a pair a_0, $b_0 \in \mathbb{C}$ such that $\operatorname{Im}(a_0 \cdot \bar{b}_0) < 0$ and $a_0N + b_0D$ is a Hurwitz matrix polynomial.*

(ii) *$\mathscr{S}_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$, where $m := \deg D$.*

(iii) *For each pair $a,\ b \in \mathbb{C}$ such that* $\mathrm{Im}(a\bar{b}) < 0$, *$aN + bD$ is a Hurwitz matrix polynomial.*

The proof of Proposition 6.4 is analogous to that in Proposition 6.3 and omitted.

Remark 6.5. Propositions 6.3 and 6.4 also hold if (N, D) is an NLFD for G. Indeed, one can easily obtain this fact by substituting $D^{\vee}$ for D and $N^{\vee}$ for N in Propositions 6.3 and 6.4, respectively.

7 Solution of matricial Routh-Hurwitz problems in terms of the Markov parameters

In this chapter, we will deal with a matricial Routh-Hurwitz problem for matrix polynomials F, which is to determine the inertia $\gamma'(F)$. Unlike the treatment by Lerer and Tismenetsky [60], we will not adopt the Anderson-Jury Bezoutian matrix, which was introduced and studied in Chapter 5, as a representing tool for this problem. Our decision is partly caused by the complexity of the construction of a required triple of matrix polynomials in connection to the Bezoutian matrix. Another reason is that the representation of the Bezoutian matrix via the matrix coefficients of the associating matrix polynomials (see (5.3)) is a little inconvenient.

Instead, our solutions to this problem are given in terms of the right and left Markov parameters of F. Both dual types are matricial generalizations of Markov parameters of scalar polynomials (see Gantmacher [37, Chapter XV, Section 15]), of which the right variant is firstly adopted by Choque Rivero [17, Definition 2.10]. For the treatment of these Markov parameters, the study of the Anderson-Jury Bezoutian matrix in Chapter 5 will be invoked.

One of the central idea here is to study the interrelations between the even part $F_{\langle e\rangle}$ and the odd part $F_{\langle o\rangle}$ (see Definition 7.1) of F. Both dual types of Markov parameters of F are derived from the SMP of a transfer function matrix formed by $F_{\langle e\rangle}$ and $F_{\langle o\rangle}$. We construct a matrix polynomial L related to F, which will also emerge to be a combination of $F_{\langle e\rangle}$ and $F_{\langle o\rangle}$. The basic instructment for this chapter will be an application of Proposition 6.1 to L, which induces an expression of $\gamma'(F)$ via the Markov parameters of F.

In order to give the reader an explicit explanation of the main problem and the related tools, some terminology is introduced below.

Definition 7.1. Let $n \in \mathbb{N}$ and let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be as in (2.1). Suppose that $m \in \mathbb{N}$ is chosen such that $n = 2m$ or $n = 2m-1$. Let $F_{\langle e\rangle} : \mathbb{C} \to \mathbb{C}^{p\times p}$ and $F_{\langle o\rangle} : \mathbb{C} \to \mathbb{C}^{p\times p}$ be, respectively, defined by

$$F_{\langle e\rangle}(z) := \begin{cases} \sum_{k=0}^{m} A_{2k} z^{m-k}, & \text{if } n = 2m, \\ \sum_{k=1}^{m} A_{2k-1} z^{m-k}, & \text{if } n = 2m-1, \end{cases} \tag{7.1}$$

and

$$F_{\langle o\rangle}(z) := \begin{cases} \sum_{k=1}^{m} A_{2k-1} z^{m-k}, & \text{if } n = 2m, \\ \sum_{k=1}^{m} A_{2k-2} z^{m-k}, & \text{if } n = 2m-1. \end{cases} \tag{7.2}$$

Then we call $F_{\langle e\rangle}$ (resp. $F_{\langle o\rangle}$) the *even part* (resp. the *odd part*) *of* F. It is easy to see that for $z \in \mathbb{C}$,

$$F(z) = F_{\langle e\rangle}(z^2) + zF_{\langle o\rangle}(z^2), \tag{7.3}$$

$$F_{\langle e\rangle}(z^2) = \frac{F(z) + F(-z)}{2}, \tag{7.4}$$

$$F_{\langle o\rangle}(z^2) = \frac{F(z) - F(-z)}{2z}. \tag{7.5}$$

Definition 7.2. Let $n \in \mathbb{N}$ and let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic. Then let $F_{\langle e\rangle}$ and $F_{\langle o\rangle}$ be as in (7.1) and (7.2), respectively. Let $F_\Delta(z) := zF_{\langle o\rangle}(z)$, $z \in \mathbb{C}$.

(i) Suppose that $n = 2m$. Let $\mathscr{S} := ((-1)^k s_k)_{k=0}^\infty \in \mathbb{C}_\infty^{p\times p}$, where $(s_k)_{k=0}^\infty$ is the SMP of $F_{\langle o\rangle} \cdot (F_{\langle e\rangle})^{-1}$ (resp. $(F_{\langle e\rangle})^{-1} \cdot F_{\langle o\rangle}$). We will call such $\mathscr{S}$ the *extended sequence of right* (resp. *left*) *Markov parameters* (or short *SRMP* (resp. *SLMP*)) of F.

(ii) Suppose that $n = 2m - 1$. Let $\mathscr{S} := ((-1)^k s_k)_{k=0}^\infty \in \mathbb{C}_\infty^{p\times p}$, where $(s_k)_{k=0}^\infty$ is the SMP of $F_{\langle e\rangle} \cdot (F_\Delta)^{-1}$ (resp. $(F_\Delta)^{-1} \cdot F_{\langle e\rangle}$). We will call such $\mathscr{S}$ the *extended sequence of right* (resp. *left*) *Markov parameters* (or short *SRMP* (resp. *SLMP*)) of F.

Moreover, let $k \in \mathbb{N}_0$. Then we call $\mathscr{S}_{\langle k\rangle}$ (see (3.2)) the *k-th SRMP* (resp. *k-th SLMP*) *of* F if $\mathscr{S}$ is the SRMP (resp. SLMP) of F.

Obviously one can see that

Remark 7.3. Let $n \in \mathbb{N}$ and $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic. Let $\mathscr{S} := (s_j)_{j\in\mathbb{N}_0} \in \mathbb{C}_\infty^{p\times p}$ and let $\widetilde{\mathscr{S}} := (\widetilde{s}_j)_{j\in\mathbb{N}_0} \in \mathbb{C}_\infty^{p\times p}$ such that $\widetilde{s}_j = s_j^*$ for $j \in \mathbb{Z}_{0,n-1}$. The sequence $\mathscr{S}$ is the SLMP of F if and only if $\widetilde{\mathscr{S}}$ is the SRMP of $F^\vee$.

Remark 7.4. Let $k, l \in \mathbb{N}_0$ such that $k \leq l$ and let $\mathscr{S} \in \mathbb{C}_\infty^{p\times p}$. Moreover, let $F \in \mathcal{P}_{p\times p,\mathbb{C}}$. If $\mathscr{S}_{\langle l\rangle}$ is the l-th SRMP (resp. SLMP) of F, then $\mathscr{S}_{\langle k\rangle}$ is the k-th SRMP (resp. SLMP) of F.

This chapter is mainly concerned with the following problem:

The Routh-Hurwitz problem for monic matrix polynomials in terms of their Markov parameters: Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}_{\infty,n-1,H}^{p\times p}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic with the $(n-1)$-th SRMP (resp. SLMP) $\mathscr{S}_{\langle n-1\rangle}$. Determine $\gamma'(F)$ in terms of $\mathscr{S}_{\langle n-1\rangle}$.

We remark that we generally set the $(n-1)$th SRMP or SLMP of F to be a Hermitian matrix sequence. This requirement can be viewed as a natural matricial extension of the truncated Markov parameters of a real polynomial, which is used to concern the zero localization problem for real polynomials (see Gantmacher [37, Sections 15–16]).

Before providing a solution to this problem, we will first characterize some connection between monic matrix polynomials and their SRMP (resp. SLMP). We note here that Δ is as in (3.1).

Proposition 7.5. *Let $n \in \mathbb{N}$ and let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic. Let $\mathscr{S} := (s_j)_{j\in\mathbb{N}_0} \in \mathbb{C}_\infty^{p\times p}$ and let $\widetilde{\mathscr{S}} := ((-1)^j s_j)_{j\in\mathbb{N}_0} \in \mathbb{C}_\infty^{p\times p}$.*

(i) *Suppose that $n = 1$. Let $F(z) := zI_p + s_0$ for each $z \in \mathbb{C}$. Then $(s_0, 0_p, \ldots)$ is the unique SRMP (resp. SLMP) of F.*

(ii) *Suppose that $n = 2m$ for $m \in \mathbb{N}$ and $k \in \mathbb{Z}_{n-1,\infty}$. $\mathscr{S}_{\langle k\rangle}$ is the k-th SRMP of F if and only if $F_{\langle o\rangle}$ is right $\widetilde{\mathscr{S}}$-associated with $F_{\langle e\rangle}$ and for $l \in \mathbb{Z}_{1,k-(2m-2)}$,*

$$\mathbf{H}_{m-1}^{\langle l\rangle}(\widetilde{\mathscr{S}}) = \mathbf{H}_{m-1}(\widetilde{\mathscr{S}})(\mathbf{C}_{F_{\langle e\rangle}}^{\langle 2\rangle})^l. \tag{7.6}$$

Moreover, $\mathscr{S}$ is the SRMP of F if and only if $F_{\langle o\rangle}$ is right $\widetilde{\mathscr{S}}$-associated with $F_{\langle e\rangle}$ and (7.6) holds for $l \in \mathbb{N}$.

(iii) *Suppose that $n = 2m-1$ for $m \in \mathbb{Z}_{2,\infty}$ and $k \in \mathbb{Z}_{n-1,\infty}$. $\mathscr{S}_{\langle k\rangle}$ is the k-th SRMP of F if and only if $F_{\langle e\rangle} - s_0 F_{\langle o\rangle}$ is right $\Delta\widetilde{\mathscr{S}}$-associated with $F_{\langle o\rangle}$ and for $l \in \mathbb{Z}_{2,k-(2m-4)}$,*

$$\mathbf{H}_{m-2}^{\langle l\rangle}(\widetilde{\mathscr{S}}) = \mathbf{H}_{m-2}^{\langle 1\rangle}(\widetilde{\mathscr{S}})(\mathbf{C}_{F_{\langle o\rangle}}^{\langle 2\rangle})^{l-1}. \tag{7.7}$$

Moreover, $\mathscr{S}$ is the SRMP of F if and only if $F_{\langle e\rangle} - s_0 F_{\langle o\rangle}$ is right $\widetilde{\mathscr{S}}$-associated with $F_{\langle e\rangle}$ and (7.7) holds for $l \in \mathbb{Z}_{2,\infty}$.

(iv) *Suppose that $n = 2m$ for $m \in \mathbb{N}$ and $k \in \mathbb{Z}_{n-1,\infty}$. $\mathscr{S}_{\langle k\rangle}$ is the k-th SLMP of F if and only if $F_{\langle o\rangle}$ is left $\widetilde{\mathscr{S}}$-associated with $F_{\langle e\rangle}$ and for $l \in \mathbb{Z}_{1,k-(2m-2)}$,*

$$\mathbf{H}_{m-1}^{\langle l\rangle}(\widetilde{\mathscr{S}}) = (\mathbf{C}_{F_{\langle e\rangle}}^{\langle 1\rangle})^l \mathbf{H}_{m-1}(\widetilde{\mathscr{S}}). \tag{7.8}$$

Moreover, $\mathscr{S}$ is the SLMP of F if and only if $F_{\langle o\rangle}$ is left $\widetilde{\mathscr{S}}$-associated with $F_{\langle e\rangle}$ and (7.8) holds for $l \in \mathbb{N}$.

(v) *Suppose that $n = 2m-1$ for $m \in \mathbb{Z}_{2,\infty}$ and $k \in \mathbb{Z}_{n-1,\infty}$. $\mathscr{S}_{\langle k\rangle}$ is the k-th SLMP of F if and only if $F_{\langle e\rangle} - F_{\langle o\rangle} \cdot s_0$ is left $\Delta\widetilde{\mathscr{S}}$-associated with $F_{\langle o\rangle}$ and for $l \in \mathbb{Z}_{2,k-(2m-4)}$,*

$$\mathbf{H}_{m-2}^{\langle l\rangle}(\widetilde{\mathscr{S}}) = (\mathbf{C}_{F_{\langle o\rangle}}^{\langle 1\rangle})^{l-1} \mathbf{H}_{m-2}^{\langle 1\rangle}(\widetilde{\mathscr{S}}). \tag{7.9}$$

Moreover, $\mathscr{S}$ is the SLMP of F if and only if $F_{\langle e\rangle} - F_{\langle o\rangle} \cdot s_0$ is left $\Delta\widetilde{\mathscr{S}}$-associated with $F_{\langle e\rangle}$ and (7.9) holds for $l \in \mathbb{Z}_{2,\infty}$.

Proof. The proof of (i) is trivial and thus omitted.

The proof of (ii): First we will show that $\mathscr{S}$ is the SRMP of F if and only if $F_{\langle o\rangle}$ is right $\widetilde{\mathscr{S}}$-associated with $F_{\langle e\rangle}$ and (7.6) holds for $l \in \mathbb{N}$.

Let $F_{\langle o\rangle}$ be right $\widetilde{\mathscr{S}}$-associated with $F_{\langle e\rangle}$ and let (7.6) hold for $l \in \mathbb{N}$. Further, let $G \in \mathscr{T}^{\diamond}_{p,p}$ be such that $\widetilde{\mathscr{S}}$ is the SMP of G. It follows from Proposition 4.13 (i) that $(F_{\langle o\rangle}, F_{\langle e\rangle})$ is an NRFD for G. Hence $\mathscr{S}$ is the SRMP of F.

Conversely, suppose that $\mathscr{S}$ is the SRMP of F. For each $z \in \mathbb{C}$, let $G \in \mathcal{T}^{\diamond}_{p,p}$ be given by, for each $z \in \mathbb{C}$,

$$G(z) := F_{\langle o\rangle}(z)(F_{\langle e\rangle}(z))^{-1}$$

such that $(F_{\langle o\rangle}, F_{\langle e\rangle})$ is an NRFD for G. Then $\widetilde{\mathscr{S}} \in \mathbb{C}^{p\times p}_{\infty}$ is the SMP of G. By applying Proposition 4.13 (ii), one can see that $F_{\langle o\rangle}$ is right $\widetilde{\mathscr{S}}$-associated with $F_{\langle e\rangle}$ and (7.6) holds for $l \in \mathbb{N}$.

Analogously, we can show that for each $k \in \mathbb{Z}_{n-1,\infty}$, $\mathscr{S}_{\langle k\rangle}$ is the k-th SRMP of F if and only if $F_{\langle o\rangle}$ is right $\widetilde{\mathscr{S}}$-associated with $F_{\langle e\rangle}$ and (7.6) holds for $l \in \mathbb{Z}_{1,k-(2m-2)}$.

The proof of (iii): First we will show that $\mathscr{S}$ is the SRMP of F if and only if $F_{\langle e\rangle} - s_0 F_{\langle o\rangle}$ is right $\widetilde{\mathscr{S}}$-associated with $F_{\langle e\rangle}$ and (7.7) holds for $l \in \mathbb{Z}_{2,\infty}$.

Suppose that $F_{\langle e\rangle} - s_0 F_{\langle o\rangle}$ is right $\Delta\widetilde{\mathscr{S}}$-associated with $F_{\langle o\rangle}$ and (7.7) holds for $l \in \mathbb{N}$. Due to Proposition 4.13 (i), there exists a $G \in \mathcal{T}^{\diamond}_{p,p}$ given by, for each $z \in \mathbb{C}$,

$$G(z) := \left(F_{\langle e\rangle}(z) - s_0 F_{\langle o\rangle}(z)\right)\left(F_{\langle o\rangle}(z)\right)^{-1}$$

such that $(F_{\langle e\rangle} - s_0 F_{\langle o\rangle}, F_{\langle o\rangle})$ is an NRFD for G and $\Delta\widetilde{\mathscr{S}}$ is the SMP of G. It follows that

$$\begin{aligned} F_{\langle e\rangle}(z)(zF_{\langle o\rangle}(z))^{-1} &= z^{-1}\left(G(z) + s_0\right) \\ &= z^{-1}(\sum_{j=1}^{\infty}(-1)^j z^{-j} s_j + s_0) \\ &= \sum_{j=0}^{\infty}(-1)^j z^{-(j+1)} s_j. \end{aligned}$$

Hence, $\mathscr{S}$ is the SRMP of F.

Conversely, suppose that $\mathscr{S}$ is the SRMP of F. Further suppose that for each $z \in \mathbb{C}$,

$$G(z) := \left(F_{\langle e\rangle}(z) - s_0 F_{\langle o\rangle}(z)\right)\left(F_{\langle o\rangle}(z)\right)^{-1}$$

such that $(F_{\langle e\rangle} - s_0 F_{\langle o\rangle}, F_{\langle o\rangle})$ is an NRFD for G. Then

$$\begin{aligned} G(z) &= z\left(F_{\langle e\rangle}(zF_{\langle o\rangle})^{-1} - z^{-1}s_0\right) \\ &= z\left(\sum_{j=0}^{\infty}(-1)^j z^{-(j+1)} s_j - z^{-1}s_0\right) \\ &= \sum_{j=0}^{\infty}(-1)^{j+1} z^{-(j+1)} s_{j+1}. \end{aligned}$$

It follows that $\widetilde{\mathscr{S}} \in \mathbb{C}_{\infty}^{p\times p}$ is the SMP of G. By applying Proposition 4.13 (ii), one can see that $F_{\langle e\rangle} - s_0 F_{\langle o\rangle}$ is right $\widetilde{\mathscr{S}}$-associated with $F_{\langle o\rangle}$ and (7.7) holds for $l \in \mathbb{N}$.

Analogously, we can show that for each $k \in \mathbb{Z}_{n-1,\infty}$, $\mathscr{S}_{\langle k\rangle}$ is the k-th SRMP of F if and only if $F_{\langle e\rangle} - s_0 F_{\langle o\rangle}$ is right $\widetilde{\mathscr{S}}$-associated with $F_{\langle o\rangle}$ and (7.7) holds for $l \in \mathbb{Z}_{1,k-(2m-2)}$.

The verification of (iv) (resp. (v)) is directly due to (ii) (resp. (iii)) and Remark 7.3. ☐

Proposition 7.6. *Let $n \in \mathbb{Z}_{2,\infty}$ and let $\mathscr{S} \in \mathbb{C}_{\infty}^{p\times p}$ such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}_{p,n-1}^{>}$. Then there exists a unique monic matrix polynomial $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ such that $\mathscr{S}_{\langle n-1\rangle}$ is the $(n-1)$-th SRMP (SLMP, resp.) of F.*

Proof. Suppose that $n = 2m$. $\mathscr{S}_{\langle 2m-1\rangle} \in \mathcal{K}_{p,2m-1}^{>}$ implies that $\mathscr{S}_{\langle 2m-2\rangle} \in \mathcal{H}_{p,2m-2}^{>}$. Then $\mathbf{H}_{m-1}(\mathscr{S})$ is positive definite and so is $\mathbf{H}_{m-1}(\widetilde{\mathscr{S}})$ due to the fact that $\mathbf{H}_{m-1}(\widetilde{\mathscr{S}}) = \mathcal{I}_j \mathbf{H}_{m-1}(\mathscr{S})\mathcal{I}_j$. In this case, there exists a unique monic matrix polynomial $F_1 \in \mathcal{P}_{p\times p,\mathbb{C}}$ such that

$$\mathbf{H}_{m-1}^{\langle l\rangle}(\widetilde{\mathscr{S}}) = \mathbf{H}_{m-1}(\widetilde{\mathscr{S}})\mathbf{C}_{F_1}^{\langle 2\rangle}$$

and a unique matrix polynomial $F_2 \in \mathcal{P}_{p\times p,\mathbb{C}}$ such that F_2 is right $\widetilde{\mathscr{S}}$-associated with F_1. Suppose that for each $z \in \mathbb{C}$,

$$F(z) := F_1(z^2) + zF_2(z^2).$$

According to (ii) of Proposition 7.5, F is the unique monic matrix polynomial within the set of $\mathcal{P}_{p\times p,2m,\mathbb{C}}$ such that $\mathscr{S}_{\langle 2m-1\rangle}$ is the $(2m-1)$-th SRMP of F.

Suppose that $n = 2m-1$. $\mathscr{S}_{\langle 2m-2\rangle} \in \mathcal{K}_{p,2m-2}^{>}$ implies that $\mathscr{S}_{\langle 2m-4\rangle}^{\langle 1\rangle} \in \mathcal{H}_{p,2m-4}^{>}$. Then $\mathbf{H}_{m-2}^{\langle 1\rangle}(\mathscr{S})$ is positive definite and so is $\mathbf{H}_{m-2}^{\langle 1\rangle}(\widetilde{\mathscr{S}})$ due to the fact that

$$\mathbf{H}_{m-2}^{\langle 1\rangle}(\widetilde{\mathscr{S}}) = -\mathcal{I}_j \mathbf{H}_{m-2}^{\langle 1\rangle}(\mathscr{S})\mathcal{I}_j,$$

where

$$\mathcal{I}_j := \operatorname{diag}(\underbrace{I_p, -I_p, \ldots, (-1)^{j-1}I_p}_{j}). \tag{7.10}$$

In this case, there exists a unique monic matrix polynomial $F_1 \in \mathcal{P}_{p\times p,\mathbb{C}}$ such that

$$\mathbf{H}_{m-2}^{\langle 2\rangle}(\widetilde{\mathscr{S}}) = \mathbf{H}_{m-2}^{\langle 1\rangle}(\widetilde{\mathscr{S}})\mathbf{C}_{F_1}^{\langle 2\rangle}$$

and a unique matrix polynomial $F_2 \in \mathcal{P}_{p\times p,\mathbb{C}}$ such that F_2 is right $\Delta\widetilde{\mathscr{S}}$-associated with F_1. Suppose that for each $z \in \mathbb{C}$,

$$F(z) := (1+z)F_1(z^2) + s_0 F_2(z^2)$$

Hence F is determined via $F_{\langle o\rangle}(z) = F_1(z)$ and $F_{\langle e\rangle}(z) = F_1(z) + s_0 F_2(z)$ for $z \in \mathbb{C}$. According to (iii) of Proposition 7.5, F is the unique monic matrix polynomial within the set of $\mathcal{P}_{p\times p,2m-1,\mathbb{C}}$ such that $\mathscr{S}_{\langle 2m-2\rangle}$ is the $(2m-2)$-th SRMP of F.

Analogously, we can prove that there exists a unique monic matrix polynomial $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ such that $\mathscr{S}_{\langle n-1\rangle}$ is the $(n-1)$-th SLMP of F. □

Given a sequence of monic matrix polynomials of even degree, we establish its connection to the MROSMP.

Proposition 7.7. *Let p, $m \in \mathbb{N}_0$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,2m-1,H}$. Let $k \in \mathbb{Z}_{0,2m}$ and let $F_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$. Further let $P_{1,0}(z) := I_p$ and let, for each $j \in \mathbb{Z}_{1,m}$,*

$$
\begin{aligned}
P_{1,j}(z) &:= (-1)^j (F_{2j})_{\langle e\rangle}(-z),\\
Q_{1,j}(z) &:= (-1)^{j-1} (F_{2j})_{\langle o\rangle}(-z).
\end{aligned}
$$

Then $\mathscr{S}_{\langle 2j-1\rangle}$ is the $(2j-1)$-th SRMP of F_{2j} for $j \in \mathbb{Z}_{1,m}$ if and only if both following conditions are satisfied:

(i) *$(P_{1,j})_{j=0}^m$ is a MROSMP of order m with respect to $\mathscr{S}$.*

(ii) *$Q_{1,j}$ is right $\mathscr{S}$-associated with respect to $P_{1,j}$, $j \in \mathbb{Z}_{0,m}$.*

Proof. In the following, we denote $\mathcal{I}_j$ as in (7.10) and

$$
F_{2j} := \sum_{l=0}^{2j} A_l^{(2j)} z^{2j-l}
$$

for each $j \in \mathbb{Z}_{1,m}$.

The proof for "only if" implication:

(i) For each $j \in \mathbb{Z}_{1,m}$,

$$
\begin{aligned}
\mathbf{H}^{\langle 1\rangle}_{j-1}(\mathscr{S}) &= -\mathcal{I}_j \mathbf{H}^{\langle 1\rangle}_{j-1}(\widetilde{\mathscr{S}})\mathcal{I}_j = -\mathcal{I}_j\left(\mathbf{H}_{j-1}(\widetilde{\mathscr{S}})\mathbf{C}^{\langle 2\rangle}_{(F_{2j})_{\langle e\rangle}}\right)\mathcal{I}_j\\
&= \left(\mathcal{I}_j \mathbf{H}_{j-1}(\widetilde{\mathscr{S}})\mathcal{I}_j\right)\cdot\left(-\mathcal{I}_j \mathbf{C}^{\langle 2\rangle}_{(F_{2j})_{\langle e\rangle}}\mathcal{I}_j\right)\\
&= \mathbf{H}_{j-1}(\mathscr{S})\cdot \mathbf{C}^{\langle 2\rangle}_{P_{1,j}},
\end{aligned}
$$

where the 2nd equality comes from (ii) of Proposition 7.5. Using Proposition 4.24 we can see that $(P_{1,j})_{j=0}^m$ is a MROSMP of order m with respect to $\mathscr{S}$.

(ii) For each $j \in \mathbb{Z}_{1,m}$,

$$
\begin{aligned}
Q_{1,j}(z) &= (-1)^{j-1}(F_{2j})_{\langle o\rangle}(-z) \\
&= ((-1)^{j-1}I_p, (-1)^{j-2}zI_p, \dots, z^{j-1}I_p)\mathcal{S}^{\langle I\rangle}_{j-1}(\widetilde{\mathscr{S}})\begin{pmatrix} A^{(2j)}_{2j-2} \\ A^{(2j)}_{2j-4} \\ \vdots \\ A^{(2j)}_{0} \end{pmatrix} \\
&= (I_p, zI_p, \dots, z^{j-1}I_p)\mathcal{I}_j\mathcal{S}^{\langle I\rangle}_{j-1}(\widetilde{\mathscr{S}})\mathcal{I}_j\begin{pmatrix} (-1)^{j-1}A^{(2j)}_{2j-2} \\ (-1)^{j-2}A^{(2j)}_{2j-4} \\ \vdots \\ A^{(2j)}_{0} \end{pmatrix} \\
&= (I_p, zI_p, \dots, z^{j-1}I_p)\mathcal{S}^{\langle I\rangle}_{j-1}(\mathscr{S})\begin{pmatrix} (-1)^{j-1}A^{(2j)}_{2j-2} \\ (-1)^{j-2}A^{(2j)}_{2j-4} \\ \vdots \\ A^{(2j)}_{0} \end{pmatrix},
\end{aligned}
$$

where the 2nd equality comes from (ii) of Proposition 7.5. This implies that $Q_{1,j}$ is right $\mathscr{S}$-associated with respect to $P_{1,j}$.

The proof for "if" implication: Using statement (i) and Proposition 4.24, we have for each $j \in \mathbb{Z}_{1,m}$,

$$
\begin{aligned}
\mathbf{H}^{\langle 1\rangle}_{j-1}(\widetilde{\mathscr{S}}) &= -\mathcal{I}_j\mathbf{H}^{\langle 1\rangle}_{j-1}(\mathscr{S})\mathcal{I}_j = -\mathcal{I}_j\left(\mathbf{H}_{j-1}(\mathscr{S})\cdot\mathbf{C}^{\langle 2\rangle}_{P_{1,j}}\right)\mathcal{I}_j \\
&= (\mathcal{I}_j\mathbf{H}_{j-1}(\mathscr{S})\mathcal{I}_j)\cdot\left(-\mathcal{I}_j\mathbf{C}^{\langle 2\rangle}_{P_{1,j}}\mathcal{I}_j\right) \\
&= \mathbf{H}_{j-1}(\widetilde{\mathscr{S}})\mathbf{C}^{\langle 2\rangle}_{(F_{2j})_{\langle e\rangle}}.
\end{aligned}
\tag{7.11}
$$

By applying statement (ii) and Proposition 4.24, one can see for each $j \in \mathbb{Z}_{1,m}$,

$$
\begin{aligned}
(F_{2j})_{\langle o\rangle}(z) &= (-1)^{j-1} Q_{1,j}(-z) \\
&= (-1)^{j-1}(I_p, (-z)I_p, \dots, (-z)^{j-1}I_p)\mathcal{S}^{\langle I\rangle}_{j-1}(\mathscr{S}) \begin{pmatrix} (-1)^{j-1}A^{(2j)}_{2j-2} \\ (-1)^{j-2}A^{(2j)}_{2j-4} \\ \vdots \\ A^{(2j)}_0 \end{pmatrix} \\
&= (I_p, zI_p, \dots, z^{j-1}I_p)\mathcal{I}_j \mathcal{S}^{\langle I\rangle}_{j-1}(\mathscr{S})\mathcal{I}_j \begin{pmatrix} A^{(2j)}_{2j-2} \\ A^{(2j)}_{2j-4} \\ \vdots \\ A^{(2j)}_0 \end{pmatrix} \\
&= (I_p, zI_p, \dots, z^{j-1}I_p)\mathcal{S}^{\langle I\rangle}_{j-1}(\widetilde{\mathscr{S}}) \begin{pmatrix} A^{(2j)}_{2j-2} \\ A^{(2j)}_{2j-4} \\ \vdots \\ A^{(2j)}_0 \end{pmatrix},
\end{aligned}
$$

which implies that $(F_{2j})_{\langle o\rangle}$ is right $\widetilde{\mathscr{S}}$-associated with respect to $(F_{2j})_{\langle e\rangle}$. With a combination of (7.11) and (ii) of Proposition 7.5, it follows that $\mathscr{S}_{\langle 2j-1\rangle}$ is the $(2j-1)$-th SRMP of F_{2j} for each $j \in \mathbb{Z}_{1,m}$. $\square$

Next we establish the dual connection between a sequence of monic matrix polynomials of odd degree and the MROSMP.

Proposition 7.8. *Let p, $m \in \mathbb{N}_0$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,2m,H}$. Let $k \in \mathbb{Z}_{0,2m+1}$ and let $F_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$. Further, for each $j \in \mathbb{Z}_{1,m+1}$, let*

$$
\begin{aligned}
P_{2,j}(z) &:= (-1)^{j-1}(F_{2j-1})_{\langle o\rangle}(-z), \\
Q_{2,j}(z) &:= (-1)^{j-1}(F_{2j-1})_{\langle e\rangle}(-z).
\end{aligned}
$$

Then $\mathscr{S}_{\langle 2j-2\rangle}$ is the $(2j-2)$-th SRMP of F_{2j-1} for $j \in \mathbb{Z}_{1,m+1}$ if and only if both following conditions are satisfied:

(i) *$(P_{2,j})^{m+1}_{j=1}$ is a MROSMP of order $(m-1)$ with respect to $\Delta\mathscr{S}$.*

(ii) *$Q_{2,j} - \mathscr{S}_{\langle 0\rangle}P_{2,j}$ is right $\Delta\mathscr{S}$-associated with respect to $P_{2,j}$, $j \in \mathbb{Z}_{1,m+1}$.*

Proof. In the following we denote for each $j \in \mathbb{Z}_{0,m}$,

$$
\mathcal{I}_{j+1} := \operatorname{diag}(\underbrace{I_p, -I_p, \dots, (-1)^j I_p}_{j+1})
$$

and

$$F_{2j+1} := \sum_{l=0}^{2j+1} A_l^{(2j+1)} z^{2j+1-l}.$$

(i) By using (iii) of Proposition 7.5, we obtain for $j \in \mathbb{Z}_{1,m+1}$,

$$\begin{aligned}\mathbf{H}_{j-1}^{\langle 2\rangle}(\mathscr{S}) &= \mathcal{I}_j \mathbf{H}_{j-1}^{\langle 2\rangle}(\widetilde{\mathscr{S}})\mathcal{I}_j = \mathcal{I}_j \left(\mathbf{H}_{j-1}^{\langle 1\rangle}(\widetilde{\mathscr{S}})\mathbf{C}_{(F_{2j})_{\langle o\rangle}}^{\langle 2\rangle}\right)\mathcal{I}_j \\ &= \left(-\mathcal{I}_j \mathbf{H}_{j-1}^{\langle 1\rangle}(\widetilde{\mathscr{S}})\mathcal{I}_j\right) \cdot \left(-\mathcal{I}_j \mathbf{C}_{(F_{2j})_{\langle o\rangle}}^{\langle 2\rangle} \mathcal{I}_j\right) \\ &= \mathbf{H}_{j-1}^{\langle 1\rangle}(\mathscr{S}) \cdot \mathbf{C}_{P_{2,j}}^{\langle 2\rangle}.\end{aligned}$$

(ii) According to (iv) of Proposition 7.5, one can see for $j \in \mathbb{Z}_{1,m+1}$,

$$\begin{aligned}Q_{2,j}(z) - \mathscr{S}_{\langle 0\rangle} P_{2,j}(z) &= (-1)^{j-1}\left((F_{2j-1})_{\langle e\rangle}(-z) - \mathscr{S}_{\langle 0\rangle}(F_{2j-1})_{\langle o\rangle}(-z)\right) \\ &= (-1)^{j-1}\left(I_p, -zI_p, \ldots, (-z)^{j-1} I_p\right) \mathcal{S}_{j-1}^{\langle I\rangle}(\widetilde{\mathscr{S}}) \begin{pmatrix} A_{2j-4}^{(2j-1)} \\ \vdots \\ A_0^{(2j-1)} \end{pmatrix} \\ &= (I_p, zI_p, \ldots, z^{j-1} I_p)\mathcal{I}_j \mathcal{S}_{j-1}^{\langle I\rangle}(\widetilde{\mathscr{S}})\mathcal{I}_j \begin{pmatrix} (-1)^{j-1} A_{2j-2}^{(2j-1)} \\ (-1)^{j-2} A_{2j-4}^{(2j-1)} \\ \vdots \\ A_0^{(2j-1)} \end{pmatrix} \\ &= (I_p, zI_p, \ldots, z^{j-1} I_p)\mathcal{S}_{j-1}^{\langle I\rangle}(\mathscr{S}) \begin{pmatrix} (-1)^{j-1} A_{2j-2}^{(2j-1)} \\ (-1)^{j-2} A_{2j-4}^{(2j-1)} \\ \vdots \\ A_0^{(2j-1)} \end{pmatrix},\end{aligned}$$

which implies that $Q_{2,j} - \mathscr{S}_{\langle 0\rangle} P_{2,j}$ is right $\Delta\mathscr{S}$-associated with respect to $P_{2,j}$, $j \in \mathbb{Z}_{1,[\frac{n+1}{2}]}$.

The proof for "if" implication: Using statement (i) and Proposition 4.24, we have for each $j \in \mathbb{Z}_{1,[\frac{n+1}{2}]}$,

$$\begin{aligned}\mathbf{H}_{j-1}^{\langle 2\rangle}(\widetilde{\mathscr{S}}) &= \mathcal{I}_j \mathbf{H}_{j-1}^{\langle 2\rangle}(\mathscr{S})\mathcal{I}_j = \mathcal{I}_j \left(\mathbf{H}_{j-1}^{\langle 1\rangle}(\mathscr{S}) \cdot \mathbf{C}_{P_{2,j}}^{\langle 2\rangle}\right)\mathcal{I}_j \\ &= \left(-\mathcal{I}_j \mathbf{H}_{j-1}^{\langle 1\rangle}(\mathscr{S})\mathcal{I}_j\right) \cdot \left(-\mathcal{I}_j \mathbf{C}_{P_{2,j}}^{\langle 2\rangle} \mathcal{I}_j\right) \\ &= \mathbf{H}_{j-1}^{\langle 1\rangle}(\widetilde{\mathscr{S}})\mathbf{C}_{(F_{2j})_{\langle o\rangle}}^{\langle 2\rangle}.\end{aligned}$$

By applying statement (ii) and Proposition 4.24, one can see for each $j \in \mathbb{Z}_{1,[\frac{n+1}{2}]}$,

$$\begin{aligned}
&(F_{2j-1})_{\langle e\rangle}(z) - \mathscr{S}_{\langle 0\rangle}(F_{2j-1})_{\langle o\rangle}(z)\\
=&(-1)^{j-1}\Big(Q_{2,j}(-z) - \mathscr{S}_{\langle 0\rangle}P_{2,j}(-z)\Big)\\
=&(-1)^{j-1}\Big(I_p, -zI_p, \dots, (-z)^{j-1}I_p\Big)\mathcal{S}^{\langle I\rangle}_{j-1}(\mathscr{S})\begin{pmatrix}(-1)^{j-1}A^{(2j-1)}_{2j-2}\\(-1)^{j-2}A^{(2j-1)}_{2j-4}\\\vdots\\A^{(2j-1)}_{0}\end{pmatrix}\\
=&(I_p, zI_p, \dots, z^{j-1}I_p)\mathcal{I}_j\mathcal{S}^{\langle I\rangle}_{j-1}(\mathscr{S})\mathcal{I}_j\begin{pmatrix}A^{(2j-1)}_{2j-2}\\A^{(2j-1)}_{2j-4}\\\vdots\\A^{(2j-1)}_{0}\end{pmatrix}\\
=&(I_p, zI_p, \dots, z^{j-1}I_p)\mathcal{S}^{\langle I\rangle}_{j-1}(\widetilde{\mathscr{S}})\begin{pmatrix}A^{(2j-1)}_{2j-2}\\\vdots\\A^{(2j-1)}_{0}\end{pmatrix},
\end{aligned}$$

which implies that $(F_{2j-1})_{\langle e\rangle} - s_0(F_{2j-1})_{\langle o\rangle}$ is right $\widetilde{\mathscr{S}}$-associated with respect to $(F_{2j-1})_{\langle o\rangle}$. With a combination of (7.11) and (ii) of Proposition 7.5, it follows that $\mathscr{S}_{\langle 2j-2\rangle}$ is the $(2j-2)$-th SRMP of F_{2j-1} for each $j \in \mathbb{Z}_{1,[\frac{n+1}{2}]}$. □

A combination of Propositions 7.7 and 7.8 gives that

Proposition 7.9. *Let p, $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$. Let $k \in \mathbb{Z}_{0,n}$ and let $F_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$. Further let*

$$\begin{aligned}
P_{1,j}(z) &:= (-1)^j(F_{2j})_{\langle e\rangle}(-z), \quad j \in \mathbb{Z}_{0,[\frac{n}{2}]},\\
P_{2,j}(z) &:= (-1)^{j-1}(F_{2j-1})_{\langle o\rangle}(-z), \quad j \in \mathbb{Z}_{1,[\frac{n+1}{2}]},\\
Q_{1,j}(z) &:= (-1)^{j-1}(F_{2j})_{\langle o\rangle}(-z), \quad j \in \mathbb{Z}_{0,[\frac{n}{2}]},\\
Q_{2,j}(z) &:= (-1)^{j-1}(F_{2j-1})_{\langle e\rangle}(-z), \quad j \in \mathbb{Z}_{1,[\frac{n+1}{2}]}.
\end{aligned}$$

Then $\mathscr{S}_{\langle k-1\rangle}$ is the $(k-1)$-th SRMP of F_k for $k \in \mathbb{Z}_{1,n}$ if and only if

(i) $(P_{1,j})_{j=0}^{[\frac{n}{2}]}$ *is a MROSMP of order $[\frac{n}{2}]$ with respect to $\mathscr{S}$.*

(ii) $Q_{1,j}$ *is right $\mathscr{S}$-associated with respect to $P_{1,j}$, $j \in \mathbb{Z}_{0,[\frac{n}{2}]}$.*

(iii) $(P_{2,j})_{j=1}^{[\frac{n+1}{2}]}$ *is a MROSMP of order $[\frac{n-1}{2}]$ with respect to $\Delta\mathscr{S}$.*

(iv) $Q_{2,j} - \mathscr{S}_{\langle 0 \rangle} P_{2,j}$ *is right* $\Delta\mathscr{S}$*-associated with respect to* $P_{2,j}$, $j \in \mathbb{Z}_{1,[\frac{n+1}{2}]}$.

By substituting the sequences $(F_j^{\vee})_{j=0}^{2m}$, $(P_{1,j}^{\vee})_{j=0}^{m}$ and $(Q_{1,j}^{\vee})_{j=0}^{m}$ for the corresponding sequences $(F_j)_{j=0}^{2m}$, $(P_{1,j})_{j=0}^{m}$ and $(Q_{1,j})_{j=0}^{m}$ in Proposition 7.7, respectively, we find that $\mathscr{S}_{\langle 2j-1 \rangle}$ is the $(2j-1)$-th SRMP of $F_{2j}^{\vee}$ for $j \in \mathbb{Z}_{1,m}$ if and only if

(i) $(P_{1,j}^{\vee})_{j=0}^{m}$ is a MROSMP of order m with respect to $\mathscr{S}$.

(ii) $Q_{1,j}^{\vee}$ is right $\mathscr{S}$-associated with respect to $P_{1,j}^{\vee}$, $j \in \mathbb{Z}_{1,m}$.

Then applying Remarks 7.3, 4.22 and 4.12 gives rise to the following proposition.

Proposition 7.10. *Let* p, $m \in \mathbb{N}_0$ *and let* $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,2m-1,H}$. *Let* $k \in \mathbb{Z}_{0,2m}$ *and let* $F_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$. *Further let* $P_{1,0}(z) := I_p$ *and let, for each* $j \in \mathbb{Z}_{1,m}$,

$$P_{1,j}(z) := (-1)^j (F_{2j})_{\langle e \rangle}(-z),$$
$$Q_{1,j}(z) := (-1)^{j-1} (F_{2j})_{\langle o \rangle}(-z).$$

Then $\mathscr{S}_{\langle 2j-1 \rangle}$ *is the* $(2j-1)$*-th SLMP of* F_{2j} *for* $j \in \mathbb{Z}_{1,m}$ *if and only if*

(i) $(P_{1,j})_{j=0}^{m}$ *is a MLOSMP of order* m *with respect to* $\mathscr{S}$.

(ii) $Q_{1,j}$ *is left* $\mathscr{S}$*-associated with respect to* $P_{1,j}$, $j \in \mathbb{Z}_{0,m}$.

Analogously one can derive the following two propositions from Propositions 7.8 and 7.9, respectively.

Proposition 7.11. *Let* p, $m \in \mathbb{N}_0$ *and let* $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,2m,H}$. *Let* $k \in \mathbb{Z}_{0,2m+1}$ *and let* $F_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$. *Further let, for each* $j \in \mathbb{Z}_{1,m+1}$

$$P_{2,j}(z) := (-1)^{j-1} (F_{2j-1})_{\langle o \rangle}(-z),$$
$$Q_{2,j}(z) := (-1)^{j-1} (F_{2j-1})_{\langle e \rangle}(-z).$$

Then $\mathscr{S}_{\langle 2j-2 \rangle}$ *is the* $(2j-2)$*-th SLMP of* F_{2j-1} *for* $j \in \mathbb{Z}_{1,m+1}$ *if and only if both following conditions are satisfied:*

(i) $(P_{2,j})_{j=1}^{m+1}$ *is a MLOSMP of order* $(m-1)$ *with respect to* $\Delta\mathscr{S}$.

(ii) $Q_{2,j} - P_{2,j}\mathscr{S}_{\langle 0 \rangle}$ *is left* $\Delta\mathscr{S}$*-associated with respect to* $P_{2,j}$, $j \in \mathbb{Z}_{1,m+1}$.

Proposition 7.12. *Let* p, $n \in \mathbb{N}$ *and let* $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$. *Let* $k \in \mathbb{Z}_{0,n}$ *and let* $F_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$. *Further let*

$$P_{1,j}(z) := (-1)^j (F_{2j})_{\langle e \rangle}(-z), \quad j \in \mathbb{Z}_{0,[\frac{n}{2}]},$$
$$P_{2,j}(z) := (-1)^{j-1} (F_{2j-1})_{\langle o \rangle}(-z), \quad j \in \mathbb{Z}_{1,[\frac{n+1}{2}]},$$
$$Q_{1,j}(z) := (-1)^{j-1} (F_{2j})_{\langle o \rangle}(-z), \quad j \in \mathbb{Z}_{0,[\frac{n}{2}]},$$
$$Q_{2,j}(z) := (-1)^{j-1} (F_{2j-1})_{\langle e \rangle}(-z), \quad j \in \mathbb{Z}_{1,[\frac{n+1}{2}]}.$$

Then $\mathscr{S}_{\langle k-1 \rangle}$ *is the* $(k-1)$*-th SLMP of* F_k *for* $k \in \mathbb{Z}_{1,n}$ *if and only if all of the following conditions are satisfied:*

(i) $(P_{1,j})_{j=0}^{[\frac{n}{2}]}$ *is a MLOSMP of order* $[\frac{n}{2}]$ *with respect to* $\mathscr{S}$.

(ii) $Q_{1,j}$ *is left* $\mathscr{S}$*-associated with respect to* $P_{1,j}$, $j \in \mathbb{Z}_{0,[\frac{n}{2}]}$.

(iii) $(P_{2,j})_{j=1}^{[\frac{n+1}{2}]}$ *is a MLOSMP of order* $[\frac{n-1}{2}]$ *with respect to* $\Delta\mathscr{S}$.

(iv) $Q_{2,j} - P_{2,j}\mathscr{S}_{\langle 0\rangle}$ *is left* $\Delta\mathscr{S}$*-associated with respect to* $P_{2,j}$, $j \in \mathbb{Z}_{1,[\frac{n+1}{2}]}$.

Definition 7.13. Let $p, m \in \mathbb{N} \cup \infty$ and let $\mathscr{S} \in \mathbb{C}_{\infty,2m-1,H}^{p\times p}$ such that $\mathscr{S}_{\langle 2m-2\rangle} \in \mathcal{H}_{p,2m-2}^{>}$. Let $(P_k^{\langle L\rangle})_{k=0}^{m}$ (resp. $(P_k^{\langle R\rangle})_{k=0}^{m}$) be the unique *MLOSMP* (resp. *MROSMP*) *of order* m *with respect to* $\mathscr{S}$. Then we say that $(Q_k^{\langle L\rangle})_{k=0}^{m}$ (resp. $(Q_k^{\langle R\rangle})_{k=0}^{m}$) is a *left* (resp. *right*) *system of matrix polynomials of the second kind* (*LSMPSK*) (resp. *RSMPSK*) *of order* m *with respect to* $\mathscr{S}$ if $Q_k^{\langle L\rangle}$ (resp. $Q_k^{\langle R\rangle}$) is left (resp. right) $\mathscr{S}$-associated with respect to $P_k^{\langle L\rangle}$ (resp. $P_k^{\langle R\rangle}$) for $k \in \mathbb{Z}_{0,m}$.

Remark 7.14. Consider a more special case that $\mathscr{S} \in \mathcal{K}_{p,n-1}^{>}$. If $n = 2m$ ($m \in \mathbb{N}$), then $(s_j)_{j=0}^{2m-2} \in \mathcal{H}_{p,2m-2}^{>}$ and $(s_{j+1})_{j=0}^{2m-2} \in \mathcal{H}_{p,2m-2}^{>}$. By using Remark 4.25, Definition 7.13 and Propositions 7.9–7.12, we have that $\mathscr{S}_{\langle k-1\rangle}$ is the $(k-1)$-th SRMP (resp. SLMP) of F_k for $k \in \mathbb{Z}_{1,n}$ if and only if all of the following conditions are satisfied:

(i) $(P_{1,j})_{j=0}^{m}$ is the unique MROSMP (resp. MLOSMP) of order m with respect to $\mathscr{S}$.

(ii) $(Q_{1,j})_{j=0}^{m}$ is the unique RSMPSK (resp. LSMPSK) of order m with respect to $\mathscr{S}$.

(iii) $(P_{2,j})_{j=1}^{m}$ is the unique MROSMP (resp. MLOSMP) of order $(m-1)$ with respect to $\Delta\mathscr{S}$.

(iv) $(Q_{2,j} - \mathscr{S}_{\langle 0\rangle}P_{2,j})_{j=1}^{m}$ (resp. $(Q_{2,j} - P_{2,j}\mathscr{S}_{\langle 0\rangle})_{j=1}^{m}$) is the unique RSMPSK (resp. LSMPSK) of order $(m-1)$ with respect to $\Delta\mathscr{S}$.

Analogously, if $n = 2m-1$ ($m \in \mathbb{N}$), then $(s_j)_{j=0}^{2m-2} \in \mathcal{H}_{p,2m-2}^{>}$ and $(s_{j+1})_{j=0}^{2m-4} \in \mathcal{H}_{p,2m-4}^{>}$. By using Remark 4.25, Definition 7.13 and Propositions 7.9–7.12, we have that $\mathscr{S}_{\langle k-1\rangle}$ is the $(k-1)$-th SRMP (resp. SLMP) of F_k for $k \in \mathbb{Z}_{1,n}$ if and only if all of the following conditions are satisfied:

(i) $(P_{1,j})_{j=0}^{m-1}$ is the unique MROSMP (resp. MLOSMP) of order $(m-1)$ with respect to $\mathscr{S}$.

(ii) $(Q_{1,j})_{j=0}^{m-1}$ is the unique RSMPSK (resp. LSMPSK) of order $(m-1)$ with respect to $\mathscr{S}$.

(iii) $(P_{2,j})_{j=1}^{m}$ is the unique MROSMP (resp. MLOSMP) of order $(m-1)$ with respect to $\Delta\mathscr{S}$.

(iv) $(Q_{2,j} - \mathscr{S}_{\langle 0 \rangle} P_{2,j})_{j=1}^{m}$ (resp. $(Q_{2,j} - P_{2,j}\mathscr{S}_{\langle 0 \rangle})_{j=1}^{m}$) is the unique RSMPSK (resp. LSMPSK) of order $(m-1)$ with respect to $\Delta\mathscr{S}$.

Now we turn back to the Routh-Hurwitz problem for monic matrix polynomials F of degree n in terms of the SRMP or SLMP of F.

For the case that $n = 1$, one can easily see that

Proposition 7.15. *Let $s_0 \in \mathbb{C}_H^{p\times p}$.*

(i) *Let $F \in \mathcal{P}_{p\times p,1,\mathbb{C}}$ be monic with the SRMP $(s_0, 0_p, \ldots)$. Then $\gamma'_-(F) = \pi(s_0)$, $\gamma'_+(F) = \nu(s_0)$, and $\gamma'_0(F) = \delta(s_0)$.*

(ii) *Let $F \in \mathcal{P}_{p\times p,1,\mathbb{C}}$ be monic with the SLMP $(s_0, 0_p, \ldots)$. Then $\gamma'_-(F) = \pi(s_0)$, $\gamma'_+(F) = \nu(s_0)$, and $\gamma'_0(F) = \delta(s_0)$.*

Proof. Use (i) of Proposition 7.5. □

Next we consider the more general case that $n \in \mathbb{Z}_{2,\infty}$.

Theorem 7.16. *Let $n \in \mathbb{Z}_{2,\infty}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic and let $\mathscr{S}_{\langle n-1 \rangle}$ be the $(n-1)$-th SRMP of F. Then*

$$\begin{aligned}
\gamma'_-(F) =& \pi(\mathbf{H}_{[\frac{n-1}{2}]}(\mathscr{S})) + \pi(\mathbf{H}^{\langle 1 \rangle}_{[\frac{n}{2}]-1}(\mathscr{S})) + \gamma_+(F^\diamond),\\
\gamma'_+(F) =& \nu(\mathbf{H}_{[\frac{n-1}{2}]}(\mathscr{S})) + \nu(\mathbf{H}^{\langle 1 \rangle}_{[\frac{n}{2}]-1}(\mathscr{S})) + \gamma_-(F^\diamond),\\
\gamma'_0(F) =& \delta(\mathbf{H}_{[\frac{n-1}{2}]}(\mathscr{S})) + \delta(\mathbf{H}^{\langle 1 \rangle}_{[\frac{n}{2}]-1}(\mathscr{S})) - \gamma_+(F^\diamond) - \gamma_-(F^\diamond),
\end{aligned}$$

where $\widehat{F}_{\langle e \rangle}(z) := F_{\langle e \rangle}(-z^2)$, $\widehat{F}_{\langle o \rangle}(z) := zF_{\langle o \rangle}(-z^2)$ for $z \in \mathbb{C}$ and $F^\diamond$ is a g.r.c.d of $\widehat{F}_{\langle e \rangle}$ and $\widehat{F}_{\langle o \rangle}$.

Proof. Let $\mathscr{S}_{\langle n-1 \rangle} := (s_k)_{k=0}^{n-1}$ and let

$$L(z) := F(\mathrm{i}z) \tag{7.12}$$

for each $z \in \mathbb{C}$. Our proof will be divided into the following two cases.

Case I: $n = 2m$. Proposition 4.15 reveals that $F_{\langle o \rangle} \cdot F_{\langle e \rangle}^{-1}$ is Hermitian and then so is $\widehat{F}_{\langle o \rangle} \cdot (\widehat{F}_{\langle e \rangle})^{-1}$. Let $\widehat{\mathscr{S}}$ be the SMP of $\widehat{F}_{\langle o \rangle} \cdot (\widehat{F}_{\langle e \rangle})^{-1}$. Then

$$\mathbf{H}_{2m-1}(\widehat{\mathscr{S}}) = \begin{pmatrix}
s_0 & 0_p & s_1 & \cdots & s_{m-1} & 0_p\\
0_p & s_1 & 0_p & \cdots & 0_p & s_m\\
s_1 & 0_p & s_2 & \cdots & s_m & 0_p\\
\vdots & \vdots & \vdots & & \vdots & \vdots\\
s_{m-1} & 0_p & s_m & \cdots & s_{2m-2} & 0_p\\
0_p & s_m & 0_p & \cdots & 0_p & s_{2m-1}
\end{pmatrix}.$$

By exchanging some rows and the corresponding columns of $\mathbf{H}_{2m-1}(\widehat{\mathscr{S}})$, we obtain then $\mathbf{H}_{2m-1}(\widehat{\mathscr{S}})$ is congruent to

$$\begin{pmatrix} \mathbf{H}_{m-1}(\mathscr{S}) & 0_{mp} \\ 0_{mp} & \mathbf{H}^{\langle 1\rangle}_{m-1}(\mathscr{S}) \end{pmatrix}.$$

It follows that

$$\operatorname{In}\left(\mathbf{H}_{2m-1}(\widehat{\mathscr{S}})\right) = \operatorname{In}(\mathbf{H}_{m-1}(\mathscr{S})) + \operatorname{In}(\mathbf{H}^{\langle 1\rangle}_{m-1}(\mathscr{S})). \tag{7.13}$$

By combination of Proposition 6.1 and (7.13),

$$\begin{aligned}\gamma'_-(F) =&\gamma_+(L) = \gamma_+(\widehat{F}_{\langle e\rangle} + \mathrm{i}\widehat{F}_{\langle o\rangle}) = \pi\left(\mathbf{H}_{2m-1}(\widehat{\mathscr{S}})\right) + \gamma_+(F^\diamond) \\ =&\pi(\mathbf{H}_{m-1}(\mathscr{S})) + \pi(\mathbf{H}^{\langle 1\rangle}_{m-1}(\mathscr{S})) + \gamma_+(F^\diamond).\end{aligned}$$

Analogously, we can conclude that

$$\begin{aligned}\gamma'_+(F) =&\nu(\mathbf{H}_{m-1}(\mathscr{S})) + \nu(\mathbf{H}^{\langle 1\rangle}_{m-1}(\mathscr{S})) + \gamma_-(F^\diamond), \\ \gamma'_0(F) =&\delta(\mathbf{H}_{m-1}(\mathscr{S})) + \delta(\mathbf{H}^{\langle 1\rangle}_{m-1}(\mathscr{S})) - \gamma_+(F^\diamond) - \gamma_-(F^\diamond),\end{aligned}$$

Hence Theorem 7.16 is verified for Case I.

Case II: $n = 2m-1$. Proposition 4.15 reveals that $F_{\langle e\rangle} \cdot F^{-1}_{\langle o\rangle}$ is Hermitian and then so is $\widehat{F}_{\langle e\rangle} \cdot (\widehat{F}_{\langle o\rangle})^{-1}$. Let $\widehat{\mathscr{S}}$ be the SMP of $\widehat{F}_{\langle e\rangle} \cdot (\widehat{F}_{\langle o\rangle})^{-1}$. Then

$$\mathbf{H}_{2m-2}(\widehat{\mathscr{S}}) = \begin{pmatrix} s_0 & 0 & s_1 & \cdots & s_{m-1} \\ 0 & s_1 & 0 & \cdots & 0 \\ s_1 & 0 & s_2 & \cdots & s_m \\ \vdots & \vdots & \vdots & & \vdots \\ s_{m-1} & 0 & s_m & \cdots & s_{2m-2} \end{pmatrix}.$$

By exchanging some rows and the corresponding columns of $\mathbf{H}_{2m-2}(\widehat{\mathscr{S}})$, we obtain then $\mathbf{H}_{2m-2}(\widehat{\mathscr{S}})$ is congruent to

$$\begin{pmatrix} \mathbf{H}_{m-1}(\mathscr{S}) & 0_{mp} \\ 0_{mp} & \mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S}) \end{pmatrix}.$$

It follows that

$$\operatorname{In}\left(\mathbf{H}_{2m-2}(\widehat{\mathscr{S}})\right) = \operatorname{In}(\mathbf{H}_{m-1}(\mathscr{S})) + \operatorname{In}(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})). \tag{7.14}$$

By combination of Proposition 6.1 and (7.14),

$$\begin{aligned}\gamma'_-(F) =&\gamma_+(L) = \gamma_+(\widehat{F}_{\langle e\rangle} + \mathrm{i}\widehat{F}_{\langle o\rangle}) = \pi\left(\mathbf{H}_{2m-1}(\widehat{\mathscr{S}})\right) + \gamma_+(F^\diamond) \\ =&\pi(\mathbf{H}_{m-1}(\mathscr{S})) + \pi(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) + \gamma_+(F^\diamond).\end{aligned}$$

Analogously, we can conclude that

$$\begin{aligned}\gamma'_+(F) =&\nu(\mathbf{H}_{m-1}(\mathscr{S})) + \nu(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) + \gamma_-(F^\diamond),\\ \gamma'_0(F) =&\delta(\mathbf{H}_{m-1}(\mathscr{S})) + \delta(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) - \gamma_+(F^\diamond) - \gamma_-(F^\diamond).\end{aligned}$$

Hence Theorem 7.16 is verified for Case II.

□

Let $n \in \mathbb{N}_0$ and let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$. Denote that $\gamma_{(-\infty,0]}(F)$ is the number of zeros of F (counting for multiplicities) lying on the non-positive real axis $(-\infty, 0]$ (the multiplicity of infinity as a zero of F is defined to be equal to np).

Next we will represent another solution to the inertia problem of matrix polynomials, which can be viewed as a refinement of Theorem 7.16.

Theorem 7.17. *Let $n \in \mathbb{Z}_{2,\infty}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic and let $\mathscr{S}_{\langle n-1\rangle}$ be the $(n-1)$-th SRMP of F. Then*

(i) *Suppose that $n = 2m$ with $m \in \mathbb{N}$.*

$$\begin{aligned}\gamma'_-(F) =&\pi(\mathbf{H}_{m-1}(\mathscr{S})) + \pi(\mathbf{H}^{\langle 1\rangle}_{m-1}(\mathscr{S})) + \delta(\mathbf{H}_{m-1}(\mathscr{S})) - \gamma_{(-\infty,0]}(\widetilde{F}),\\ \gamma'_+(F) =&\nu(\mathbf{H}_{m-1}(\mathscr{S})) + \nu(\mathbf{H}^{\langle 1\rangle}_{m-1}(\mathscr{S})) + \delta(\mathbf{H}_{m-1}(\mathscr{S})) - \gamma_{(-\infty,0]}(\widetilde{F}),\\ \gamma'_0(F) =&\delta(\mathbf{H}^{\langle 1\rangle}_{m-1}(\mathscr{S})) - \delta(\mathbf{H}_{m-1}(\mathscr{S})) + 2\gamma_{(-\infty,0]}(\widetilde{F}),\end{aligned}$$

where $\widetilde{F}$ is a g.r.c.d of $F_{\langle e\rangle}$ and $F_{\langle o\rangle}$.

(ii) *Suppose that $n = 2m-1$ with $m \in \mathbb{N}$. Then*

$$\begin{aligned}\gamma'_-(F) =&\pi(\mathbf{H}_{m-1}(\mathscr{S})) + \pi(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) + \delta(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) - \gamma_{(-\infty,0]}(\widetilde{F}),\\ \gamma'_+(F) =&\nu(\mathbf{H}_{m-1}(\mathscr{S})) + \nu(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) + \delta(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) - \gamma_{(-\infty,0]}(\widetilde{F}),\\ \gamma'_0(F) =&\delta(\mathbf{H}_{m-1}(\mathscr{S})) - \delta(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) + 2\gamma_{(-\infty,0]}(\widetilde{F}),\end{aligned}$$

where $\widetilde{F}$ is a g.r.c.d of $F_{\langle e\rangle}$ and $F_{\langle o\rangle}$.

Proof. Let $\widehat{F}_{\langle e\rangle}$, $\widehat{F}_{\langle o\rangle}$ and $F^\diamond$ be as in Theorem 7.16. Let $\widetilde{F}_{\langle o\rangle}(z) := F_{\langle o\rangle}(-z^2)$ and $\widehat{F}(z) := \widetilde{F}(-z^2)$ for each $z \in \mathbb{C}$. Then by applying Proposition 2.26, we obtain that $\widehat{F}$ is a g.r.c.d of $\widehat{F}_{\langle e\rangle}$ and $\widetilde{F}_{\langle o\rangle}$. By Proposition 2.20 it follows that

$$\sigma(\widehat{F}) \cup \{0\} = \sigma(F^\diamond) \cup \{0\}.$$

Hence

$$\gamma_+(F^\diamond) = \gamma_+(\widehat{F}),\ \gamma_-(F^\diamond) = \gamma_-(\widehat{F}). \tag{7.15}$$

By Remark 2.23 we see that $\lambda \in \sigma(\widehat{F})$ if and only if $-\lambda \in \sigma(\widehat{F})$. Consequently

$$\gamma_+(\widehat{F}) = \gamma_-(\widehat{F}). \tag{7.16}$$

Let the two mappings $\phi_+ : \mathbb{C} \to \mathbb{C}$ and $\phi_- : \mathbb{C} \to \mathbb{C}$ be defined, for each $z \in \mathbb{C}$, by

$$\phi_+(z) := |z|^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}(\frac{1}{2}\mathrm{Arg}(z)+\frac{\pi}{2})}, \quad \phi_-(z) := -|z|^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}(\frac{1}{2}\mathrm{Arg}(z)+\frac{\pi}{2})}.$$

Without difficulty we can see that both ϕ_+ and ϕ_- are bijective and

$$\phi_+^{-1}(z) = \phi_-^{-1}(z) = -z^2$$

for each $z \in \mathbb{C}$.

Adopting Remark 2.23 again yields that $\lambda \in \sigma(\widetilde{F})$ if and only if $\phi_+(\lambda) \in \sigma(\widehat{F})$ if and only if $\phi_-(\lambda) \in \sigma(\widehat{F})$. Then

$$\gamma_0(\widehat{F}) = 2\gamma_{(-\infty,0]}(\widetilde{F}). \tag{7.17}$$

Our next discussion will be divided into the following two cases.

Case I: $n = 2m$. Noticing that $\mathbf{H}_{m-1}(\mathscr{S})$ is a Hermitian matrix, we get

$$\mathrm{dimKer}\mathbf{H}_{m-1}(\mathscr{S}) = \delta(\mathbf{H}_{m-1}(\mathscr{S})). \tag{7.18}$$

Accordingly, we obtain that

$$\begin{aligned}\deg(\det\widehat{F}(z)) &= 2\deg(\det\widetilde{F}(z)) = 2\mathrm{dimKer}(\mathbf{B}_{F_{\langle e\rangle}^{\vee}, F_{\langle o\rangle}^{\vee}}(F_{\langle e\rangle}, F_{\langle o\rangle})) \\ &= 2\mathrm{dimKer}(\mathbf{H}_{m-1}(\mathscr{S})) = 2\delta(\mathbf{H}_{m-1}(\mathscr{S})),\end{aligned} \tag{7.19}$$

where the 1st equation is due to Remark 2.23, the 2nd equation is due to Proposition 5.14, the 3rd equation is due to Proposition 5.12 and the last equation is due to (7.18).

In view of (7.16), (7.17) and (7.19), we conclude that

$$\begin{aligned}\gamma_+(F^\diamond) = \gamma_+(\widehat{F}) &= \frac{1}{2}(\deg(\det\widehat{F}(z)) - \gamma_0(\widehat{F})) = \frac{1}{2}\deg(\det\widehat{F}(z)) - \gamma_{(-\infty,0]}(\widetilde{F}) \\ &= \delta(\mathbf{H}_{m-1}(\mathscr{S})) - \gamma_{(-\infty,0]}(\widetilde{F}).\end{aligned} \tag{7.20}$$

By combination of Theorem 7.16, (7.15), (7.16) and (7.20), Theorem 7.17 is verified for Case I.

Case II: $n = 2m-1$. Noticing that $\mathbf{H}_{m-2}(\mathscr{S})$ is a Hermitian matrix, we get

$$\mathrm{dimKer}\mathbf{H}_{m-2}^{\langle 1\rangle}(\mathscr{S}) = \delta(\mathbf{H}_{m-2}^{\langle 1\rangle}(\mathscr{S})). \tag{7.21}$$

Then it follows that

$$\begin{aligned}\deg(\det\widehat{F}(z)) &= 2\deg(\det\widetilde{F}(z)) = 2\mathrm{dimKer}(\mathbf{B}_{F_{\langle o\rangle}^{\vee}, F_{\langle e\rangle}^{\vee} - F_{\langle o\rangle}^{\vee}\cdot\mathscr{S}_{\langle 0\rangle}}(F_{\langle o\rangle}, F_{\langle e\rangle} - \mathscr{S}_{\langle 0\rangle}\cdot F_{\langle o\rangle})) \\ &= 2\mathrm{dimKer}(\mathbf{H}_{m-2}^{\langle 1\rangle}(\mathscr{S})) = 2\delta(\mathbf{H}_{m-2}^{\langle 1\rangle}(\mathscr{S})),\end{aligned} \tag{7.22}$$

where the 1st equation is due to Remark 2.23, the 2nd equation is due to Propositions 2.17 and 5.14, the 3rd equation is due to Proposition 5.12 and the last equation is due to (7.21).

In view of (7.16), (7.17) and (7.22), we show that

$$\begin{aligned}\gamma_+(F^\diamond) = \gamma_+(\widehat{F}) = \frac{1}{2}(\deg(\det\widehat{F}(z)) - \gamma_0(\widehat{F})) = \frac{1}{2}\deg(\det\widehat{F}(z)) - \gamma_{(-\infty,0]}(\widetilde{F})\\ = \delta(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) - \gamma_{(-\infty,0]}(\widetilde{F}).\end{aligned} \tag{7.23}$$

By combination of Theorem 7.16, (7.15), (7.16) and (7.23), Theorem 7.17 is verified for Case II. □

Theorem 7.18. *Let $n \in \mathbb{Z}_{2,\infty}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic with an $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1\rangle}$. Then*

$$\begin{aligned}\gamma'_-(F) =&\pi(\mathbf{H}_{[\frac{n-1}{2}]}(\mathscr{S})) + \pi(\mathbf{H}^{\langle 1\rangle}_{[\frac{n}{2}]-1}(\mathscr{S})) + \gamma_-(F^\diamond),\\ \gamma'_+(F) =&\nu(\mathbf{H}_{[\frac{n-1}{2}]}(\mathscr{S})) + \nu(\mathbf{H}^{\langle 1\rangle}_{[\frac{n}{2}]-1}(\mathscr{S})) + \gamma_+(F^\diamond),\\ \gamma'_0(F) =&\delta(\mathbf{H}_{[\frac{n-1}{2}]}(\mathscr{S})) + \delta(\mathbf{H}^{\langle 1\rangle}_{[\frac{n}{2}]-1}(\mathscr{S})) - \gamma_+(F^\diamond) - \gamma_-(F^\diamond),\end{aligned}$$

where $\widehat{F}_{\langle e\rangle}(z) := F_{\langle e\rangle}(-z^2)$, $\widehat{F}_{\langle o\rangle}(z) := zF_{\langle o\rangle}(-z^2)$ for $z \in \mathbb{C}$ and $F^\diamond$ is a g.l.c.d of $\widehat{F}_{\langle e\rangle}$ and $\widehat{F}_{\langle o\rangle}$.

Proof. Applying Remark 7.3 gives that $\mathscr{S}_{\langle n-1\rangle}$ is the $(n-1)$-th SRMP of $F^\vee$. According to Theorem 7.16, we obtain that

$$\begin{aligned}\gamma'_-(F) = \gamma'_-(F^\vee) =&\pi(\mathbf{H}_{[\frac{n-1}{2}]}(\mathscr{S})) + \pi(\mathbf{H}^{\langle 1\rangle}_{[\frac{n}{2}]-1}(\mathscr{S})) + \gamma_+(\widetilde{F}^\diamond),\\ \gamma'_+(F) = \gamma'_+(F^\vee) =&\nu(\mathbf{H}_{[\frac{n-1}{2}]}(\mathscr{S})) + \nu(\mathbf{H}^{\langle 1\rangle}_{[\frac{n}{2}]-1}(\mathscr{S})) + \gamma_-(\widetilde{F}^\diamond),\\ \gamma'_0(F) = \gamma'_0(F^\vee) =&\delta(\mathbf{H}_{[\frac{n-1}{2}]}(\mathscr{S})) + \delta(\mathbf{H}^{\langle 1\rangle}_{[\frac{n}{2}]-1}(\mathscr{S})) - \gamma_+(\widetilde{F}^\diamond) - \gamma_-(\widetilde{F}^\diamond),\end{aligned}$$

where $\widetilde{F}^\diamond$ is a g.r.c.d of $\widehat{F}^\vee_{\langle e\rangle}$ and $\widehat{F}^\vee_{\langle o\rangle}$. Let $F^\diamond(z) := (\widetilde{F}^\diamond)^\vee(z)$ for $z \in \mathbb{C}$. By Proposition 2.18, $F^\diamond$ is a g.l.c.d of $\widehat{F}_{\langle e\rangle}$ and $\widehat{F}_{\langle o\rangle}$. By noticing the obvious statements that $\gamma_-(F^\diamond) = \gamma_+(\widetilde{F}^\diamond)$ and $\gamma_+(F^\diamond) = \gamma_-(\widetilde{F}^\diamond)$, we obtain the result of Theorem 7.18. □

Adopting the analogous proof as in the proof of Theorem 7.18, the following result comes from Theorem 7.17.

Theorem 7.19. *Let $n \in \mathbb{Z}_{2,\infty}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{n-1,\infty}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic with an $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1\rangle}$. Then*

(i) *Suppose that $n = 2m$ with $m \in \mathbb{N}$.*

$$\begin{aligned}\gamma'_-(F) =&\pi(\mathbf{H}_{m-1}(\mathscr{S})) + \pi(\mathbf{H}^{\langle 1\rangle}_{m-1}(\mathscr{S})) + \delta(\mathbf{H}_{m-1}(\mathscr{S})) - \gamma_{(-\infty,0]}(\widetilde{F}),\\ \gamma'_+(F) =&\nu(\mathbf{H}_{m-1}(\mathscr{S})) + \nu(\mathbf{H}^{\langle 1\rangle}_{m-1}(\mathscr{S})) + \delta(\mathbf{H}_{m-1}(\mathscr{S})) - \gamma_{(-\infty,0]}(\widetilde{F}),\\ \gamma'_0(F) =&\delta(\mathbf{H}^{\langle 1\rangle}_{m-1}(\mathscr{S})) - \delta(\mathbf{H}_{m-1}(\mathscr{S})) + 2\gamma_{(-\infty,0]}(\widetilde{F}),\end{aligned}$$

where $\widetilde{F}$ is a g.l.c.d of $F_{\langle e\rangle}$ and $F_{\langle o\rangle}$.

(ii) *Suppose that* $n = 2m - 1$ *with* $m \in \mathbb{N}$. *Then*

$$\begin{aligned}
\gamma'_{-}(F) =&\pi(\mathbf{H}_{m-1}(\mathscr{S})) + \pi(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) + \delta(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) - \gamma_{(-\infty,0]}(\widetilde{F}),\\
\gamma'_{+}(F) =&\nu(\mathbf{H}_{m-1}(\mathscr{S})) + \nu(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) + \delta(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) - \gamma_{(-\infty,0]}(\widetilde{F}),\\
\gamma'_{0}(F) =&\delta(\mathbf{H}_{m-1}(\mathscr{S})) - \delta(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) + 2\gamma_{(-\infty,0]}(\widetilde{F}),
\end{aligned}$$

where $\widetilde{F}$ *is a g.l.c.d of* $F_{\langle e\rangle}$ *and* $F_{\langle o\rangle}$.

8 Matrix Hurwitz type polynomials and some related topics

In this chapter we will pay our attention to right and left matrix Hurwitz type polynomials, which are characterized by certain Stieltjes matrix continuous fractions. The reason of Choque Rivero [17] for introducing such a concept originates from the criterion for scalar Hurwitz polynomial in terms of a certain Stieltjes continuous fraction (see [37, Theorem 16]). The paper [17] pays attention to the question of criteria for a matrix polynomial to be a right matrix Hurwitz type polynomial. The obtained characterization of a right matrix Hurwitz type polynomial F is expressed in terms of its SRMP $\mathscr{S}$, where the right Hurwitz parametrization (see Definition 8.1) of F coincides with a particular parametrization of $\mathscr{S}$.

This chapter decides to dig a little deeper to answer inverse questions concerning a relation between Hurwitz matrix type polynomials and their right Hurwitz parametrization: Given an ordered pair of sequences of positive definite matrices $[(\mathbf{c}_j)_{j=0}^{[\frac{n}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{n-1}{2}]}]$, does there exist a unique right matrix Hurwitz type polynomial F of degree n such that $[(\mathbf{c}_j)_{j=0}^{[\frac{n}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{n-1}{2}]}]$ is the right Hurwitz parametrization of F? And if the answer is positive, what is the representation of F in terms of $[(\mathbf{c}_j)_{j=0}^{[\frac{n}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{n-1}{2}]}]$?

To solve these problems, the central idea is to firstly seek a three-term recurrence relation for a sequence of matrix Hurwitz type polynomials $(F_l)_{l=1}^{n}$ in terms of the right Hurwitz parametrizations of F_n. Then we check whether F_n is the matrix Hurwitz type polynomial which is uniquely required.

We proceed with an introduction of matrix Hurwitz type polynomials. Before doing this, we introduce a notation. Let A, $B \in \mathbb{C}^{p\times p}$ such that B is nonsingular. Denote

$$\frac{A}{B} := A \cdot B^{-1}.$$

Definition 8.1. Let $n \in \mathbb{N}$ and let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic.

(i) Suppose that $n = 2m$. Then F is called a *right matrix Hurwitz type polynomial* if there exists an ordered pair of sequences $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-1}]$ from $\mathbb{C}_{>}^{p\times p}$ such that the identity

$$\frac{F_{\langle o\rangle}(z)}{F_{\langle e\rangle}(z)} = \cfrac{I_p}{z\mathbf{c}_0 + \cfrac{I_p}{\mathbf{d}_0 + \cfrac{I_p}{\ddots \, \mathbf{d}_{m-2} + \cfrac{I_p}{z\mathbf{c}_{m-1} + \mathbf{d}_{m-1}^{-1}}}}}$$

holds for each $z \in \mathbb{C}$ such that $|z| > \max\{|\lambda| : \lambda \in \sigma(F_{\langle e\rangle})\}$. In this case the pair $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-1}]$ is called a *right Hurwitz parametrization of F*.

(ii) Suppose that $n = 2m - 1$. Then $F(z)$ is called a *right matrix Hurwitz type polynomial* if there exists an ordered pair of sequences $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-2}]$ from $\mathbb{C}_>^{p\times p}$ such that the identity

$$\frac{F_{\langle e\rangle}(z)}{zF_{\langle o\rangle}(z)} = \cfrac{I_p}{z\mathbf{c}_0 + \cfrac{I_p}{\mathbf{d}_0 + \cfrac{I_p}{\ddots \, z\mathbf{c}_{m-2} + \cfrac{I_p}{\mathbf{d}_{m-2} + z^{-1}\mathbf{c}_{m-1}^{-1}}}}}$$

holds for each $z \in \mathbb{C}$ such that $|z| > \max\{|\lambda| : \lambda \in \sigma(F_{\langle o\rangle})\}$. In this case the pair $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-2}]$ is called a *right Hurwitz parametrization of $F(z)$*.

(iii) Suppose that $n = 2m$. Then F is called a *left matrix Hurwitz type polynomial* if there exists an ordered pair of sequences $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-1}]$ from $\mathbb{C}_>^{p\times p}$ such that the identity

$$\left(F_{\langle e\rangle}(z)\right)^{-1} F_{\langle o\rangle}(z) = \cfrac{I_p}{z\mathbf{c}_0 + \cfrac{I_p}{\mathbf{d}_0 + \cfrac{I_p}{\ddots \, \mathbf{d}_{m-2} + \cfrac{I_p}{z\mathbf{c}_{m-1} + \mathbf{d}_{m-1}^{-1}}}}}$$

holds for each $z \in \mathbb{C}$ such that $|z| > \max\{|\lambda| : \lambda \in \sigma(F_{\langle e\rangle})\}$. In this case the pair $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-1}]$ is called a *left Hurwitz parametrization of F*.

(iv) Suppose that $n = 2m-1$. Then F is called a *left matrix Hurwitz type polynomial* if there exists an ordered pair of sequences $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-2}]$ from $\mathbb{C}_>^{p\times p}$ such that the identity

$$\left(zF_{\langle o\rangle}(z)\right)^{-1}F_{\langle e\rangle}(z)=\cfrac{I_p}{z\mathbf{c}_0+\cfrac{I_p}{\mathbf{d}_0+\cfrac{I_p}{\ddots\, z\mathbf{c}_{m-2}+\cfrac{}{\mathbf{d}_{m-2}+z^{-1}\mathbf{c}_{m-1}^{-1}}}}}$$

holds for each $z\in\mathbb{C}$ such that $|z|>\max\{|\lambda|:\lambda\in\sigma(F_{\langle o\rangle})\}$. In this case the pair $[(\mathbf{c}_k)_{k=0}^{m-1},(\mathbf{d}_k)_{k=0}^{m-2}]$ is called a *left Hurwitz parametrization of F*.

We mention that the two notions "right matrix Hurwitz type polynomial" and "right Hurwitz parametrization" are called "matrix Hurwitz type polynomial" and "Hurwitz parametrization" in [17], respectively.

Remark 8.2. Let $n\in\mathbb{N}$ and let $F\in\mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic. Then F is a right matrix Hurwitz type polynomial if and only if $F^{\vee}$ is a left matrix Hurwitz type polynomial. In this case, let $\{\mathbf{c}_k\}_{k=0}^{[\frac{n}{2}]}\cup\{\mathbf{d}_k\}_{k=0}^{[\frac{n-1}{2}]}\subseteq\mathbb{C}_{>}^{p\times p}$. Then $[(\mathbf{c}_k)_{k=0}^{[\frac{n}{2}]},(\mathbf{d}_k)_{k=0}^{[\frac{n-1}{2}]}]$ is a right Hurwitz parametrization of F if and only if $[(\mathbf{c}_k)_{k=0}^{[\frac{n}{2}]},(\mathbf{d}_k)_{k=0}^{[\frac{n-1}{2}]}]$ is a left Hurwitz parametrization of $F^{\vee}$.

In the following, we will build a three-term recurrence relation for a sequence of right matrix Hurwitz type polynomials (see Lemma 8.3 below). The coefficients of the recurrence are indeed related to the right Hurwitz parametrizations of these right matrix Hurwitz type polynomials. To achieve this result, we will use the method of linear fractional transformation, of which the terminology is introduced below.

Let $A\in\mathbb{C}^{2p\times 2p}$ be partitioned into the following 2-by-2 block matrices:

$$A:=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}.$$

where $A_{jk}\in\mathbb{C}^{p\times p}$ for all j, $k\in\mathbb{Z}_{1,2}$. If

$$\mathcal{D}_{\langle A_{21},A_{22}\rangle}:=\{X\in\mathbb{C}^{p\times p}:\det(A_{21}X+A_{22})\neq 0\}$$

is non-empty, then let the linear fractional transformation $\Psi_A^p\colon\mathcal{D}_{\langle A_{21},A_{22}\rangle}\to\mathbb{C}^{p\times p}$ be defined by

$$\Psi_A^p(X):=\frac{A_{11}X+A_{12}}{A_{21}X+A_{22}},\quad X\in\mathcal{D}_{A_{21},A_{22}}.$$

Let $k\in\mathbb{N}_0$ and let $\{A_j\}_{j=0}^{k}\subseteq\mathbb{C}^{p\times p}$. Then let

$$\overrightarrow{\prod}_{j=0}^{k}A_j:=A_0\cdot A_1\cdots A_k\text{ and }\overleftarrow{\prod}_{j=0}^{k}A_j:=A_k\cdot A_{k-1}\cdots A_0$$

denote the right and left product of the matrices A_0, A_1, …, A_n, respectively.

Lemma 8.3. *Let $n \in \mathbb{N} \cup \infty$. Let $\{\mathbf{c}_k\}_{k=0}^{[\frac{n}{2}]} \cup \{\mathbf{d}_k\}_{k=0}^{[\frac{n-1}{2}]} \subseteq \mathbb{C}_{>}^{p\times p}$ and let $(F_l)_{l=0}^{n}$ be related by the following recurrence form:*

$$F_{2k}(z) = zF_{2k-1}(z) + F_{2k-2}(z)\mathbf{e}_{k-2}\mathbf{e}_{k-1}^{-1}, \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]} \tag{8.1}$$

$$F_{2k+1}(z) = zF_{2k}(z) + F_{2k-1}(z)\mathbf{e}_{k-2}\mathbf{c}_{k-1}\mathbf{c}_k^{-1}\mathbf{e}_{k-1}^{-1}, \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]} \tag{8.2}$$

with the initial values

$$F_0(z) = I_p, \quad F_1(z) = zI_p + \mathbf{c}_0^{-1}, \tag{8.3}$$

where

$$\mathbf{e}_k := \begin{cases} I_p, & k = -1, \\ \overrightarrow{\prod}_{j=0}^{k}\mathbf{c}_j\mathbf{d}_j, & k \in \mathbb{N}_0. \end{cases}$$

Then for each $l \in \mathbb{Z}_{1,n}$, F_l is a right matrix Hurwitz type polynomial with the right Hurwitz parametrization $[(\mathbf{c}_j)_{j=0}^{[\frac{l}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{l-1}{2}]}]$.

Proof. By assuming that

$$\widetilde{F}_{2k+1}(z) := F_{2k+1}(z)\mathbf{e}_{k-1}\mathbf{c}_k, \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}, \tag{8.4}$$

$$\widetilde{F}_{2k}(z) := F_{2k}(z)\mathbf{e}_{k-1}, \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]}, \tag{8.5}$$

we transform (8.1)–(8.3) into the following forms:

$$\widetilde{F}_{2k}(z) = z\widetilde{F}_{2k-1}(z)\mathbf{d}_{k-1} + \widetilde{F}_{2k-2}(z), \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]}, \tag{8.6}$$

$$\widetilde{F}_{2k+1}(z) = z\widetilde{F}_{2k}(z)\mathbf{c}_k + \widetilde{F}_{2k-1}(z), \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}, \tag{8.7}$$

with the initial values

$$\widetilde{F}_0(z) = I_p, \quad \widetilde{F}_1(z) = z\mathbf{c}_0 + I_p. \tag{8.8}$$

In the case that $l = 2k+1$ (resp. $l = 2k$) with $l \in \mathbb{Z}_{0,n}$, we assume that $\widetilde{P}_l$ and $\widetilde{Q}_l$ are the even (resp. odd) part and the odd (resp. even) part of $\widetilde{F}_l$, respectively. In view of (8.6)–(8.8), we have

$$\widetilde{P}_{2k}(z) = \widetilde{P}_{2k-1}(z)\mathbf{d}_{k-1} + \widetilde{P}_{2k-2}(z), \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]}, \tag{8.9}$$

$$\widetilde{P}_{2k+1}(z) = z\widetilde{P}_{2k}(z)\mathbf{c}_k + \widetilde{P}_{2k-1}(z), \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}, \tag{8.10}$$

and

$$\widetilde{Q}_{2k}(z) = z\widetilde{Q}_{2k-1}(z)\mathbf{d}_{k-1} + \widetilde{Q}_{2k-2}(z), \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]}, \tag{8.11}$$

$$\widetilde{Q}_{2k+1}(z) = \widetilde{Q}_{2k}(z)\mathbf{c}_k + \widetilde{Q}_{2k-1}(z), \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}. \tag{8.12}$$

with the initial values

$$\widetilde{P}_0(z) = 0_p, \quad \widetilde{Q}_0(z) = I_p, \quad \widetilde{P}_1(z) = I_p, \quad \widetilde{Q}_1(z) = \mathbf{c}_0. \tag{8.13}$$

Let, for each $z \in \mathbb{C}$,

$$\mathbf{W}_{2k}(z) := \begin{pmatrix} \widetilde{P}_{2k}(z) & \widetilde{P}_{2k+1}(z) \\ \widetilde{Q}_{2k}(z) & z\widetilde{Q}_{2k+1}(z) \end{pmatrix}, \quad \mathbf{C}_k(z) := \begin{pmatrix} 0_p & I_p \\ I_p & z\mathbf{c}_k \end{pmatrix}, \quad k \in \mathbb{Z}_{0,[\frac{n-1}{2}]},$$

and

$$\mathbf{W}_{2k+1}(z) := \begin{pmatrix} \widetilde{P}_{2k+1}(z) & \widetilde{P}_{2k+2}(z) \\ z\widetilde{Q}_{2k+1}(z) & \widetilde{Q}_{2k+2}(z) \end{pmatrix}, \quad \mathbf{D}_k(z) := \begin{pmatrix} 0_p & I_p \\ I_p & \mathbf{d}_k \end{pmatrix}, \quad k \in \mathbb{Z}_{0,[\frac{n}{2}]-1}.$$

We turn (8.9)–(8.13) into the following matrix recurrence form:

$$\begin{aligned} \mathbf{W}_{2k} &= \mathbf{W}_{2k-1} \cdot \mathbf{C}_k, \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}, \\ \mathbf{W}_{2k+1} &= \mathbf{W}_{2k} \cdot \mathbf{D}_k, \quad k \in \mathbb{Z}_{0,[\frac{n}{2}]-1}, \end{aligned}$$

which implies that

$$\mathbf{W}_{2k} = \prod_{j=1}^{k-1} (\mathbf{C}_j \cdot \mathbf{D}_j) \cdot \mathbf{C}_k, \quad k \in \mathbb{Z}_{0,[\frac{n-1}{2}]},$$

and

$$\mathbf{W}_{2k+1} = \prod_{j=1}^{k} (\mathbf{C}_j \cdot \mathbf{D}_j), \quad k \in \mathbb{Z}_{0,[\frac{n}{2}]-1}.$$

By applying Proposition 1.6.2 (b) of [20], we obtain that for each $X \in \mathbb{C}^{p\times p}$,

$$\Psi^p_{\mathbf{W}_{2k}}(X) = \Psi^p_{\mathbf{C}_1}\left(\Psi^p_{\mathbf{D}_1}\left(\cdots\left(\Psi^p_{\mathbf{C}_{k-1}}\left(\Psi^p_{\mathbf{D}_{k-1}}\left(\Psi^p_{\mathbf{C}_k}(X)\right)\right)\right)\cdots\right)\right), \quad k \in \mathbb{Z}_{0,[\frac{n-1}{2}]},$$

and

$$\Psi^p_{\mathbf{W}_{2k+1}}(X) = \Psi^p_{\mathbf{C}_1}\left(\Psi^p_{\mathbf{D}_1}\left(\cdots\left(\Psi^p_{\mathbf{C}_k}\left(\Psi^p_{\mathbf{D}_k}(X)\right)\right)\cdots\right)\right), \quad k \in \mathbb{Z}_{0,[\frac{n}{2}]-1}.$$

In the case that $l = 2k+1$ (resp. $l = 2k$) with $l \in \mathbb{Z}_{0,n}$, we assume that P_l and Q_l are the even (resp. odd) part and the odd (resp. even) part of F_l, respectively. Then by (8.4) and (8.5) one can see that for each

$$\widetilde{P}_{2k+1}(z) = P_{2k+1}(z)\mathbf{e}_{k-1}\mathbf{c}_k, \; \widetilde{Q}_{2k+1}(z) = Q_{2k+1}(z)\mathbf{e}_{k-1}\mathbf{c}_k, \quad k \in \mathbb{Z}_{0,[\frac{n-1}{2}]},$$

and

$$\widetilde{P}_{2k}(z) = P_{2k}(z)\mathbf{e}_{k-1}, \; \widetilde{Q}_{2k}(z) := Q_{2k}(z)\mathbf{e}_{k-1}, \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]}.$$

It follows that for each $k \in \mathbb{Z}_{1,[\frac{n}{2}]}$,

$$\begin{aligned}\frac{Q_{2k}(z)}{P_{2k}(z)} = \frac{\widetilde{Q}_{2k}(z)}{\widetilde{P}_{2k}(z)} =&\Psi^p_{\mathbf{W}_{2k-1}}(0_p) = \Psi^p_{\mathbf{C}_1}\left(\Psi^p_{\mathbf{D}_1}\left(\cdots\left(\Psi^p_{\mathbf{C}_{k-1}}\left(\Psi^p_{\mathbf{D}_{k-1}}(0_p)\right)\right)\cdots\right)\right)\\ =&\cfrac{I_p}{z\mathbf{c}_0 + \cfrac{I_p}{\mathbf{d}_0 + \cfrac{I_p}{\ddots\, \mathbf{d}_{k-2} + \cfrac{I_p}{z\mathbf{c}_{k-1} + \mathbf{d}_{k-1}^{-1}}}}},\end{aligned}$$

and for each $k \in \mathbb{Z}_{0,[\frac{n-1}{2}]}$,

$$\begin{aligned}\frac{P_{2k+1}(z)}{zQ_{2k+1}(z)} = \frac{\widetilde{P}_{2k+1}(z)}{z\widetilde{Q}_{2k+1}(z)} =&\Psi^p_{\mathbf{W}_{2k}}(0_p) = \Psi^p_{\mathbf{C}_1}\left(\Psi^p_{\mathbf{D}_1}\left(\cdots\left(\Psi^p_{\mathbf{D}_{k-1}}\left(\Psi^p_{\mathbf{C}_k}(0_p)\right)\right)\cdots\right)\right)\\ =&\cfrac{I_p}{z\mathbf{c}_0 + \cfrac{I_p}{\mathbf{d}_0 + \cfrac{I_p}{\ddots\, z\mathbf{c}_{k-1} + \cfrac{I_p}{\mathbf{d}_{k-1} + z^{-1}\mathbf{c}_k^{-1}}}}}.\end{aligned}$$

Hence for each $l \in \mathbb{Z}_{1,n}$, F_l is a monic right matrix Hurwitz type polynomial and, for each $l \in \mathbb{Z}_{1,n}$, $[(\mathbf{c}_j)_{j=0}^{[\frac{l}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{l-1}{2}]}]$ is a right Hurwitz parametrization of F_l. $\square$

In view of Lemma 8.3 and Remark 8.2, we give the following dual result.

Lemma 8.4. *Let $n \in \mathbb{N} \cup \infty$. Let $\{\mathbf{c}_k\}_{k=0}^{[\frac{n}{2}]} \cup \{\mathbf{d}_k\}_{k=0}^{[\frac{n-1}{2}]} \subseteq \mathbb{C}^{p\times p}_{>}$ and let $(F_l)_{l=1}^{n}$ be related by the following recurrence form:*

$$F_{2k}(z) = zF_{2k-1}(z) + \widetilde{\mathbf{e}}_{k-1}^{-1}\widetilde{\mathbf{e}}_{k-2}F_{2k-2}(z), \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]}, \tag{8.14}$$

$$F_{2k+1}(z) = zF_{2k}(z) + \widetilde{\mathbf{e}}_{k-1}^{-1}\mathbf{c}_k^{-1}\mathbf{c}_{k-1}\widetilde{\mathbf{e}}_{k-2}F_{2k-1}(z), \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}, \tag{8.15}$$

with the initial values

$$F_0(z) = I_p, \quad F_1(z) = zI_p + \mathbf{c}_0^{-1}, \tag{8.16}$$

where

$$\widetilde{\mathbf{e}}_k := \begin{cases} I_p, & k = -1,\\ \overleftarrow{\prod}_{j=0}^{k}\mathbf{d}_j\mathbf{c}_j, & k \in \mathbb{N}_0.\end{cases}$$

Then for each $l \in \mathbb{Z}_{1,n}$, F_l is a left matrix Hurwitz type polynomial with the left Hurwitz parametrization $[(\mathbf{c}_j)_{j=0}^{[\frac{l}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{l-1}{2}]}]$.

According to Lemmas 8.3 and 8.4, if an ordered pair of sequences of positive definite matrices $[(\mathbf{c}_j)_{j=0}^{[\frac{n}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{n-1}{2}]}]$ is given, we can build a right (resp. left) matrix Hurwitz type polynomial F such that $[(\mathbf{c}_j)_{j=0}^{[\frac{n}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{n-1}{2}]}]$ is the right (resp. left) Hurwitz parametrization of F. However, a question of the uniqueness for this construction arises naturally: If an ordered pair of sequences of positive definite matrices $[(\mathbf{c}_j)_{j=0}^{[\frac{n}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{n-1}{2}]}]$ is given, does a unique right (resp. left) matrix Hurwitz type polynomial F built as in Lemma 8.3 (resp. Lemma 8.4) such that $[(\mathbf{c}_j)_{j=0}^{[\frac{n}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{n-1}{2}]}]$ is the right (resp. left) Hurwitz parametrization of F exists?

To answer this question, we will seek some interrelations between the Hurwitz parametrization of F and the Dyukarev-Stieltjes parametrization of a positive definite sequence initiated by Fritzsche/Kirstein/Mädler [32] (see Definition 8.5 below). The introduction of the latter concept is inspired by Yu.M. Dyukarev when he was looking for an indeterminacy criterion for the Stieltjes matrix moment problem [25]. For a detailed discussion of the Dyukarev-Stieltjes parametrization, the readers are referred to [17] and [32].

Definition 8.5. Let $n \in \mathbb{N} \cup \infty$ and let $\mathscr{S}_{n-1} := (s_j)_{j=0}^{n-1} \in \mathcal{K}_{p,n-1}^{>}$. Let

$$\mathbf{M}_k := \begin{cases} s_0^{-1}, & \text{if } k = 0, \\ (I_p, 0_{p\times kp}) \left(\mathbf{H}_k(\mathscr{S}_{n-1})\right)^{-1} \begin{pmatrix} I_p \\ 0_{kp\times p} \end{pmatrix} & \\ \qquad -(I_p, 0_{p\times(k-1)p}) \left(\mathbf{H}_{k-1}(\mathscr{S}_{n-1})\right)^{-1} \begin{pmatrix} I_p \\ 0_{(k-1)p\times p} \end{pmatrix}, & \text{if } k \in \mathbb{Z}_{1,[\frac{n}{2}]}, \end{cases}$$

for all $k \in \mathbb{Z}_{0,[\frac{n}{2}]}$, and

$$\mathbf{L}_k := \begin{cases} s_0 s_1^{-1} s_0, & \text{if } k = 0, \\ \mathbf{Z}_{0,k}(\mathscr{S}_{n-1}) \left(\mathbf{H}_k^{\langle 1\rangle}(\mathscr{S}_{n-1})\right)^{-1} \mathbf{Y}_{0,k}(\mathscr{S}_{n-1}) & \\ \qquad -\mathbf{Z}_{0,k-1}(\mathscr{S}_{n-1}) \left(\mathbf{H}_{k-1}^{\langle 1\rangle}(\mathscr{S}_{n-1})\right)^{-1} \mathbf{Y}_{0,k-1}(\mathscr{S}_{n-1}), & \text{if } k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}, \end{cases}$$

for all $k \in \mathbb{Z}_{0,[\frac{n-1}{2}]}$. Then the ordered pair $[(\mathbf{L}_k)_{k=0}^{[\frac{n-1}{2}]}, (\mathbf{M}_k)_{k=0}^{[\frac{n}{2}]}]$ is called the *Dyukarev-Stieltjes parametrization* (shortly *DS-parametrization*) of $\mathscr{S}_{n-1}$.

For $n \in \mathbb{N}\cup\infty$ and every ordered pair $[(\mathbf{L}_k)_{k=0}^{[\frac{n-1}{2}]}, (\mathbf{M}_k)_{k=0}^{[\frac{n}{2}]}]$, we will show that there exists a unique positive definite sequence $\mathscr{S}_{n-1}$ such that $[(\mathbf{L}_k)_{k=0}^{[\frac{n-1}{2}]}, (\mathbf{M}_k)_{k=0}^{[\frac{n}{2}]}]$ is the DS-parametrization of $\mathscr{S}_{n-1}$.

Lemma 8.6. *[32, Proposition 8.26] Let $n \in \mathbb{N} \cup \infty$. Let $\mathscr{S}_{n-1} := (s_j)_{j=0}^{n-1} \in \mathcal{K}_{p,n-1}^{>}$*

and let $[(\mathbf{L}_k)_{k=0}^{[\frac{n-1}{2}]}, (\mathbf{M}_k)_{k=0}^{[\frac{n}{2}]}]$ *be the DS-parametrization of* $\mathscr{S}_{n-1}$*. Then*

$$s_{2k} = \begin{cases} \mathbf{M}_0^{-1}, & \text{if } k = 0, \\ \mathbf{Z}_{k,2k-1}(\mathscr{S}_{n-1}) \left(\mathbf{H}_{k-1}(\mathscr{S}_{n-1})\right)^{-1} \mathbf{Y}_{k,2k-1}(\mathscr{S}_{n-1}) & \\ \qquad + \left(\overrightarrow{\prod}_{j=0}^{k-1} \mathbf{M}_j \mathbf{L}_j\right)^{-*} \mathbf{M}_k^{-1} \left(\overrightarrow{\prod}_{j=0}^{k-1} \mathbf{M}_j \mathbf{L}_j\right)^{-1}, & \text{if } k \in \mathbb{Z}_{1,[\frac{n}{2}]}, \end{cases}$$

and

$$s_{2k+1} = \begin{cases} \left(\mathbf{M}_0 \mathbf{L}_0\right)^{-*} \mathbf{L}_0 \left(\mathbf{M}_0 \mathbf{L}_0\right)^{-1}, & \text{if } k = 0, \\ \mathbf{Z}_{k+1,2k}(\mathscr{S}_{n-1}) \left(\mathbf{H}_{k-1}^{\langle 1 \rangle}(\mathscr{S}_{n-1})\right)^{-1} \mathbf{Y}_{k+1,2k}(\mathscr{S}_{n-1}) & \\ \qquad + \left(\overrightarrow{\prod}_{j=0}^{k} \mathbf{M}_j \mathbf{L}_j\right)^{-*} \mathbf{L}_k \left(\overrightarrow{\prod}_{j=0}^{k} \mathbf{M}_j \mathbf{L}_j\right)^{-1}, & \text{if } k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}. \end{cases}$$

The following lemma shows a coincidence between the right Hurwitz parametrization and the truncated SRMP for each right matrix Hurwitz type polynomial.

Lemma 8.7. *[17, Theorem 7.9] Let* $n \in \mathbb{N}$ *and let* $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ *be a right matrix Hurwitz type polynomial. Let* $[(\mathbf{c}_j)_{j=0}^{[\frac{n}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{n-1}{2}]}]$ *be the right Hurwitz parametrization of* F *and let* $\mathscr{S}_{n-1} \in \mathbb{C}_{n-1}^{p\times p}$ *be the* $(n-1)$*-th SRMP of* F*. Then* $[(\mathbf{d}_j)_{j=0}^{[\frac{n-1}{2}]}, (\mathbf{c}_j)_{j=0}^{[\frac{n}{2}]}]$ *is the DS-parametrization of* $\mathscr{S}_{n-1}$*.*

As an immediate consequence of Lemmas 8.6 and 8.7, the following proposition characterizes the Markov parameters of right matrix Hurwitz type polynomials in terms of their right Hurwitz parametrizations.

Proposition 8.8. *Let* $n \in \mathbb{N}$ *and let* $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ *be a right matrix Hurwitz type polynomial. Let* $[(\mathbf{c}_j)_{j=0}^{[\frac{n}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{n-1}{2}]}]$ *be the right Hurwitz parametrization of* F *and let* $\mathscr{S}_{n-1} := (s_j)_{j=0}^{n-1} \in \mathbb{C}_{n-1}^{p\times p}$ *be the* $(n-1)$*-th SRMP of* F*. Then* $(s_j)_{j=0}^{n-1}$ *is related as follows:*

$$s_{2k} = \begin{cases} \mathbf{c}_0^{-1}, & \text{if } k = 0, \\ \mathbf{Z}_{k,2k-1}(\mathscr{S}_{n-1}) \left(\mathbf{H}_{k-1}(\mathscr{S}_{n-1})\right)^{-1} \mathbf{Y}_{k,2k-1}(\mathscr{S}_{n-1}) & \\ \qquad + \left(\overrightarrow{\prod}_{j=0}^{k-1} \mathbf{c}_j \mathbf{d}_j\right)^{-*} \mathbf{c}_k^{-1} \left(\overrightarrow{\prod}_{j=0}^{k-1} \mathbf{c}_j \mathbf{d}_j\right)^{-1}, & \text{if } k \in \mathbb{Z}_{1,[\frac{n}{2}]}, \end{cases}$$

and

$$s_{2k+1} = \begin{cases} \left(\mathbf{c}_0 \mathbf{d}_0\right)^{-*} \mathbf{d}_0 \left(\mathbf{c}_0 \mathbf{d}_0\right)^{-1}, & \text{if } k = 0, \\ \mathbf{Z}_{k+1,2k}(\mathscr{S}_{n-1}) \left(\mathbf{H}_{k-1}^{\langle 1 \rangle}(\mathscr{S}_{n-1})\right)^{-1} \mathbf{Y}_{k+1,2k}(\mathscr{S}_{n-1}) & \\ \qquad + \left(\overrightarrow{\prod}_{j=0}^{k} \mathbf{c}_j \mathbf{d}_j\right)^{-*} \mathbf{d}_k \left(\overrightarrow{\prod}_{j=0}^{k} \mathbf{c}_j \mathbf{d}_j\right)^{-1}, & \text{if } k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}. \end{cases}$$

Next we show some bijective correspondences between Stieltjes positive definite sequences, matrix Hurwitz type polynomials and matrix Hurwitz type polynomials.

Lemma 8.9. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}_{\infty}^{p\times p}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic with the $(n-1)$-th SRMP $\mathscr{S}_{\langle n-1\rangle}$. The following statements are equivalent:*

(i) $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}_{p,n-1}^{>}$.

(ii) *F is a right matrix Hurwitz type polynomial.*

In this case, F is the unique right matrix Hurwitz type polynomial with the $(n-1)$-th SRMP $\mathscr{S}_{\langle n-1\rangle}$.

Proof. The equivalence between (i) and (ii) comes from [17, Theorem 7.10]. The uniqueness of F in this case comes from Proposition 7.6. □

Lemma 8.10. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}_{\infty}^{p\times p}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic with the $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1\rangle}$. The following statements are equivalent:*

(i) $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}_{p,n-1}^{>}$.

(ii) *F is a left matrix Hurwitz type polynomial.*

In this case, F is the unique left matrix Hurwitz type polynomial with the $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1\rangle}$.

Proof. Apply Remark 8.2 and Lemma 8.9. □

At the end of this chapter, we are able to answer the uniqueness question for the construction of right matrix Hurwitz type polynomials in Lemma 8.3.

Theorem 8.11. *Let $n \in \mathbb{N} \cup \infty$. Let $\{\mathbf{c}_k\}_{k=0}^{[\frac{n}{2}]} \cup \{\mathbf{d}_k\}_{k=0}^{[\frac{n-1}{2}]} \subseteq \mathbb{C}_{>}^{p\times p}$ and let $(F_l)_{l=1}^{n}$ be related as in (8.1)–(8.3). Then for each $l \in \mathbb{Z}_{1,n}$, F_l is the unique right matrix Hurwitz type polynomial with the right Hurwitz parametrization $[(\mathbf{c}_j)_{j=0}^{[\frac{l}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{l-1}{2}]}]$.*

Proof. According to Lemma 8.3, F_l is a Hurwitz matrix polynomial with the right Hurwitz parametrization $[(\mathbf{c}_j)_{j=0}^{[\frac{l}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{l-1}{2}]}]$. Then we will prove by contradiction that F_l is the unique Hurwitz matrix polynomial with the right Hurwitz parametrization $[(\mathbf{c}_j)_{j=0}^{[\frac{l}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{l-1}{2}]}]$. Suppose that there exists a $l_0 \in \mathbb{Z}_{1,l}$ and $\widetilde{F}_{l_0} \in \mathcal{P}_{p\times p,l_0,\mathbb{C}}$ such that $\widetilde{F}_{l_0} \neq F_{l_0}$ and $\widetilde{F}_{l_0}$ is a Hurwitz matrix polynomial with the right Hurwitz parametrization $[(\mathbf{c}_j)_{j=0}^{[\frac{l}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{l-1}{2}]}]$. Then suppose that $\mathscr{S}_{l_0-1}$ and $\widetilde{\mathscr{S}}_{l_0-1}$ are the (l_0-1)-th SRMP of F_{l_0} and $\widetilde{F}_{l_0}$, respectively. By Lemma 8.9 one can see that both $\mathscr{S}_{l_0-1}$ and $\widetilde{\mathscr{S}}_{l_0-1}$ are in the set $\mathcal{K}_{p,l_0-1}^{>}$. We can easily see that $\mathscr{S}_{l_0-1} \neq \widetilde{\mathscr{S}}_{l_0-1}$. (Indeed, if $\mathscr{S}_{l_0-1} = \widetilde{\mathscr{S}}_{l_0-1}$, then by using Proposition 7.6, $F_{l_0} \equiv \widetilde{F}_{l_0}$, which leads to a contradiction.) So $\mathscr{S}_{l_0-1} \neq \widetilde{\mathscr{S}}_{l_0-1}$. It contradicts Proposition 8.8. Hence for each $l \in \mathbb{Z}_{1,n}$, F_l is the unique Hurwitz matrix polynomial with the right Hurwitz parametrization $[(\mathbf{c}_j)_{j=0}^{[\frac{l}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{l-1}{2}]}]$. □

In view of Theorem 8.11 and Remark 8.2, we easily obtain a dual property of uniqueness for left matrix Hurwitz type polynomials.

Theorem 8.12. *Let $n \in \mathbb{N} \cup \infty$. Let $\{\mathbf{c}_k\}_{k=0}^{[\frac{n}{2}]} \cup \{\mathbf{d}_k\}_{k=0}^{[\frac{n-1}{2}]} \subseteq \mathbb{C}_{>}^{p\times p}$ and let $(F_l)_{l=1}^{n}$ be related as in* (8.14)–(8.16)*. Then for each $l \in \mathbb{Z}_{1,n}$, F_l is the unique left matrix Hurwitz type polynomial with the left Hurwitz parametrization $[(\mathbf{c}_j)_{j=0}^{[\frac{l}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{l-1}{2}]}]$.*

9 Hurwitz matrix polynomials and some related topics

This chapter focuses on the main objective of this thesis. Various connections are established between several classes of objects: Hurwitz matrix polynomials, Stieltjes positive definite sequences, matrix Hurwitz type polynomials and Stieltjes quadruple of sequences of left orthogonal matrix polynomials.

The lines of research are combined here. In Section 9.1, we consider what conditions must be imposed on a matrix polynomial F in order to ensure that F is a Hurwitz matrix polynomial. Regarding the scalar case, an equivalent condition to identify a Hurwitz polynomial F is that the truncated Markov parameters sequence of F is a Stieltjes positive definite sequence (see Gantmacher [37, Theorem 18]). So it is natural to propose the following matricial conjecture:

F is a Hurwitz matrix polynomial if and only if the truncated SRMP or SLMP of F is a matricial Stieltjes positive definite sequence.

Our aim now is to verify this conjecture. The validation of the conjecture is essential from another point of view: Once it is shown to be true, the equivalence between Hurwitz matrix polynomials and matrix Hurwitz type polynomials can be subsequently achieved.

Section 9.2 mainly establishes an expression of a Hurwitz matrix polynomial via a three-terms recurrence relation. For this reason, this section pays attention to a particular sequence of Hurwitz matrix polynomials $(F_k)_{k=1}^{n}$ called left (resp. right) $\mathscr{S}$-system of Hurwitz matrix polynomials. This sequence is connected to a matrix sequence $\mathscr{S}$ in the sense that each F_k is a Hurwitz matrix polynomial with the $(k-1)$-th SLMP (resp. SRMP) $\mathscr{S}_{\langle k-1 \rangle}$. By seeking the relation between the three terms F_{k-1}, F_k and F_{k+1}, every Hurwitz matrix polynomial F of degree n can be iteratively formed via the $(n-1)$-th SLMP (resp. SRMP) of F and the initial matrix polynomials F_0 and F_1. The investigation of the $\mathscr{S}$-system of Hurwitz matrix polynomials therefore yields the appropriate tools to establish the desired connections between Hurwitz matrix polynomials, matrix Hurwitz type polynomials and Stieltjes quadruple of sequences of left orthogonal matrix polynomials.

9.1 Hurwitz matrix polynomials, Stieltjes positive definite sequences and matrix Hurwitz type polynomials

This section has several important points of touch with Chapters 7 and 8.

We begin this section by considering what conditions must be imposed on the

SRMP so that the corresponding matrix polynomial is a Hurwitz matrix polynomial. It is obvious to see that a Hurwitz matrix polynomial can indeed be viewed as a matrix polynomial with special zero localization with respect to left and right half planes. Taking this fact into consideration, our subsequent approach relies largely on the solutions of matricial Routh-Hurwitz problems in terms of the SRMPs as deduced in Chapter 7. Owing to the study of SRMPs of right matrix Hurwitz type polynomials in Chapter 8, we will derive the connections between both extended types of Hurwitz polynomials by using the tools of their SRMPs or SLMPs.

Theorem 9.1. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic with the $(n-1)$-th SRMP $\mathscr{S}_{\langle n-1\rangle}$. The following statements are equivalent:*

(i) *F is a Hurwitz matrix polynomial.*

(ii) *$\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{>}_{p,n-1}$.*

In this case, F is the unique Hurwitz matrix polynomial with the $(n-1)$-th SRMP $\mathscr{S}_{\langle n-1\rangle}$.

Proof. The proof for "(ii) $\Longrightarrow$ (i)": If $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{>}_{p,n-1}$, then

$$\pi(\mathbf{H}_{[\frac{n-1}{2}]}) = [\tfrac{n+1}{2}]p \quad \text{and} \qquad \pi(\mathbf{H}^{\langle 1\rangle}_{[\frac{n-1}{2}]}) = [\tfrac{n}{2}]p.$$

Accordingly, by applying Theorem 7.16, we have

$$\gamma'_{-}(F) \geq \pi(\mathbf{H}_{[\frac{n-1}{2}]}) + \pi(\mathbf{H}^{\langle 1\rangle}_{[\frac{n}{2}]-1}) = np.$$

On the other hand,

$$\gamma'_{-}(F) \leq \deg\, \det F(z) = np.$$

Thus $\gamma'_{-}(F) = np$, which implies (i).

The proof for "(i) $\Longrightarrow$ (ii)": Assume that F is a monic Hurwitz matrix polynomial with the $(n-1)$th SRMP $\mathscr{S}_{\langle n-1\rangle}$. Then $\gamma'_{-}(F) = np$.

On the other hand, suppose that $\widehat{F}_{\langle e\rangle}(z) := F_{\langle e\rangle}(-z^2)$, $\widehat{F}_{\langle o\rangle}(z) := zF_{\langle o\rangle}(-z^2)$ for $z \in \mathbb{C}$ and $F^{\diamond}$ is a g.r.c.d of $\widehat{F}_{\langle e\rangle}$ and $\widehat{F}_{\langle o\rangle}$. We can prove that $\gamma_{+}(F^{\diamond}) = 0$ by contradiction. Indeed, suppose that $\gamma_{+}(F^{\diamond}) \neq 0$ and there exists a $z_0 \in \mathbb{C}_U$ such that $z_0 \in \sigma(F^{\diamond})$. Assume that L is as in (7.12) and $L_1(z) := F(-\mathrm{i}z)$ for $z \in \mathbb{C}$. According to Proposition 2.19, $F^{\diamond}$ is a g.r.c.d of L and L_1. Then for each zero $z_0 \in \sigma(F^{\diamond})$, then $\mathrm{i}z_0 \in \sigma(F)$, $-\mathrm{i}z_0 \in \sigma(F)$, which contradicts the fact that F is a Hurwitz matrix polynomial. Consequently, by using Theorem 7.16, we get

$$np = \pi(\mathbf{H}_{[\frac{n-1}{2}]}) + \pi(\mathbf{H}^{\langle 1\rangle}_{[\frac{n}{2}]-1}). \tag{9.1}$$

By the application of (9.1) and the statements that $\pi(\mathbf{H}_{[\frac{n-1}{2}]}) \leq [\frac{n+1}{2}]p$ and $\pi(\mathbf{H}^{\langle 1\rangle}_{[\frac{n}{2}]-1}) \leq [\frac{n}{2}]p$, (ii) is easily verified. □

An immediate application of Remark 7.3 and Theorem 9.1 gives the analogous results for matrix polynomials with their SLMPs.

Theorem 9.2. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic with the $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1\rangle}$. The following statements are equivalent:*

(i) *F is a Hurwitz matrix polynomial.*

(ii) *$\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{>}_{p,n-1}$.*

In this case, F is the unique Hurwitz matrix polynomial with the $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1\rangle}$.

Theorems 9.1 and 9.2 build two bijective correspondences between positive definite sequences and Hurwitz matrix polynomials, which can be viewed as matricial generalizations of [37, Theorem 20, Chapter XV, p. 240].

In what follows, we give some examples to check whether these matrix polynomials are Hurwitz matrix polynomials or not with the tools of Theorems 9.1 and 9.2.

Example 9.3. *Suppose that $F \in \mathcal{P}_{2\times 2,2,\mathbb{C}}$ is given as*

$$F(z) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z^2 + \begin{pmatrix} 5 & -1 \\ -1 & 7 \end{pmatrix} z + \begin{pmatrix} 26 & -5 \\ -6 & 43 \end{pmatrix}.$$

Obviously we can obtain that

$$F_{\langle e\rangle}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z + \begin{pmatrix} 26 & -5 \\ -6 & 43 \end{pmatrix} \text{ and } F_{\langle o\rangle}(z) = \begin{pmatrix} 5 & -1 \\ -1 & 7 \end{pmatrix}.$$

Then the 1-st SLMP of F is as follows:

$$s_0 = \begin{pmatrix} 5 & -1 \\ -1 & 7 \end{pmatrix} \text{ and } s_1 = \begin{pmatrix} 5 & -1 \\ -1 & 7 \end{pmatrix} \cdot \begin{pmatrix} 26 & -5 \\ -6 & 43 \end{pmatrix} = \begin{pmatrix} 136 & -68 \\ -68 & 306 \end{pmatrix}.$$

It is clear that both s_0 and s_1 is positive definite. By applying Theorem 9.2, we can see that F is a Hurwitz matrix polynomial.

Example 9.4. *Let $F \in \mathcal{P}_{2\times 2,4,\mathbb{C}}$ be given as*

$$\begin{aligned} F(z) :=& \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z^4 + \begin{pmatrix} 2 & 2-\mathrm{i} \\ 2+\mathrm{i} & 3 \end{pmatrix} z^3 + \begin{pmatrix} 23+\mathrm{i} & -2-17\mathrm{i} \\ 1+39\mathrm{i} & 20-5\mathrm{i} \end{pmatrix} z^2 \\ &+ \begin{pmatrix} 61-34\mathrm{i} & 42-71\mathrm{i} \\ 48+87\mathrm{i} & 104+62\mathrm{i} \end{pmatrix} z + \begin{pmatrix} -58 & 5+39\mathrm{i} \\ 9-71\mathrm{i} & -67 \end{pmatrix}. \end{aligned}$$

Obviously

$$F_{\langle e\rangle}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z^2 + \begin{pmatrix} 23+\mathrm{i} & -2-17\mathrm{i} \\ 1+39\mathrm{i} & 20-5\mathrm{i} \end{pmatrix} z + \begin{pmatrix} -58 & 5+39\mathrm{i} \\ 9-71\mathrm{i} & -67 \end{pmatrix}$$

and

$$F_{\langle o\rangle}(z) = \begin{pmatrix} 2 & 2-\mathrm{i} \\ 2+\mathrm{i} & 3 \end{pmatrix} z + \begin{pmatrix} 61-34\mathrm{i} & 42-71\mathrm{i} \\ 48+87\mathrm{i} & 104+62\mathrm{i} \end{pmatrix}.$$

By calculation, the 3*-rd SLMP* $\mathscr{S}_{\langle 3\rangle}$ *of* F *is as follows:*

$$\mathscr{S}_{\langle 3\rangle} = \left(\begin{pmatrix} 2 & 2-\mathrm{i} \\ 2+\mathrm{i} & 3 \end{pmatrix}, \begin{pmatrix} -2 & -1-\mathrm{i} \\ -1+\mathrm{i} & -3 \end{pmatrix}, \begin{pmatrix} 13 & 2+13\mathrm{i} \\ 2-13\mathrm{i} & 20 \end{pmatrix}, \begin{pmatrix} -210 & -\mathrm{i} \\ \mathrm{i} & -377 \end{pmatrix} \right).$$

We check whether $\mathbf{H}_1(\mathscr{S}_{\langle 3\rangle})$ *is positive definite, but* $\mathbf{H}_1^{\langle 1\rangle}(\mathscr{S}_{\langle 3\rangle})$ *is not positive definite. So an application of Theorem 9.2 shows that* F *is not a Hurwitz matrix polynomial.*

Example 9.5. *Let* $F \in \mathcal{P}_{2\times 2,3,\mathbb{C}}$ *be given as*

$$F(z) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z^3 + \begin{pmatrix} 3 & 4 \\ 4 & 8 \end{pmatrix} z^2 + \begin{pmatrix} 23-15\mathrm{i} & 33+35\mathrm{i} \\ 12-10\mathrm{i} & 17+15\mathrm{i} \end{pmatrix} z + \begin{pmatrix} 115-85\mathrm{i} & 170+165\mathrm{i} \\ 191-140\mathrm{i} & 261+260\mathrm{i} \end{pmatrix}.$$

Obviously

$$F_{\langle e\rangle}(z) = \begin{pmatrix} 3 & 4 \\ 4 & 8 \end{pmatrix} z + \begin{pmatrix} 115-85\mathrm{i} & 170+165\mathrm{i} \\ 191-140\mathrm{i} & 261+260\mathrm{i} \end{pmatrix}$$

and

$$F_{\langle o\rangle}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z + \begin{pmatrix} 23-15\mathrm{i} & 33+35\mathrm{i} \\ 12-10\mathrm{i} & 17+15\mathrm{i} \end{pmatrix}.$$

By calculation, the 2*-nd SRMP of* F *is as follows:*

$$s_0 = \begin{pmatrix} 3 & 4 \\ 4 & 8 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 2 & -3 \\ -3 & 7 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 10 & 15+25\mathrm{i} \\ 15-25\mathrm{i} & 4 \end{pmatrix}.$$

We can see that s_0 *is positive definite. However,*

$$s_2 - s_1 s_0^{-1} s_1 = \begin{pmatrix} -\frac{27}{8} & \frac{323}{8}+25\mathrm{i} \\ \frac{323}{8}-25\mathrm{i} & -\frac{227}{8} \end{pmatrix},$$

which is not a positive definite matrix. So $\mathbf{H}_2(\mathscr{S})$ *is not positive definite. Applying Theorem 9.1, we show that* F *is not a Hurwitz matrix polynomial.*

Example 9.6. *Let* $F \in \mathcal{P}_{3\times 3,3,\mathbb{C}}$ *be given as*

$$\begin{aligned} F(z) := {} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} z^3 + \begin{pmatrix} 2 & -1-\mathrm{i} & \mathrm{i} \\ -1+\mathrm{i} & 2 & -1 \\ -\mathrm{i} & -1 & 2 \end{pmatrix} z^2 + \begin{pmatrix} 13-\mathrm{i} & 8+3\mathrm{i} & 9+4\mathrm{i} \\ 1+6\mathrm{i} & -1+4\mathrm{i} & -2+5\mathrm{i} \\ -4\mathrm{i} & 1-3\mathrm{i} & 3-3\mathrm{i} \end{pmatrix} z \\ & + \begin{pmatrix} 34-9\mathrm{i} & 24+3\mathrm{i} & 28+9\mathrm{i} \\ -10+31\mathrm{i} & -16+16\mathrm{i} & -20+18\mathrm{i} \\ -2-28\mathrm{i} & 6-18\mathrm{i} & 9-20\mathrm{i} \end{pmatrix}. \end{aligned}$$

Obviously

$$F_{\langle e\rangle}(z) = \begin{pmatrix} 2 & -1-\mathrm{i} & \mathrm{i} \\ -1+\mathrm{i} & 2 & -1 \\ -\mathrm{i} & -1 & 2 \end{pmatrix} z + \begin{pmatrix} 34-9\mathrm{i} & 24+3\mathrm{i} & 28+9\mathrm{i} \\ -10+31\mathrm{i} & -16+16\mathrm{i} & -20+18\mathrm{i} \\ -2-28\mathrm{i} & 6-18\mathrm{i} & 9-20\mathrm{i} \end{pmatrix}$$

and

$$F_{\langle o\rangle}(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} z + \begin{pmatrix} 13-\mathrm{i} & 8+3\mathrm{i} & 9+4\mathrm{i} \\ 1+6\mathrm{i} & -1+4\mathrm{i} & -2+5\mathrm{i} \\ -4\mathrm{i} & 1-3\mathrm{i} & 3-3\mathrm{i} \end{pmatrix}.$$

By calculation, the 2-nd SRMP of F is as follows:

$$s_0 = \begin{pmatrix} 2 & -1-\mathrm{i} & \mathrm{i} \\ -1+\mathrm{i} & 2 & -1 \\ -\mathrm{i} & -1 & 2 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 1 & \mathrm{i} & -\mathrm{i} \\ -\mathrm{i} & 2 & 0 \\ \mathrm{i} & 0 & 3 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 15 & 1+5\mathrm{i} & 3-5\mathrm{i} \\ 1-5\mathrm{i} & 10 & -6\mathrm{i} \\ 3+5\mathrm{i} & 6\mathrm{i} & 50 \end{pmatrix}.$$

We can see that both s_0 and s_1 are positive definite and

$$s_2 - s_1 s_0^{-1} s_1 = \begin{pmatrix} 10 & -\frac{3}{2}-\frac{1}{2}\mathrm{i} & -\frac{1}{2}+\frac{1}{2}\mathrm{i} \\ -\frac{3}{2}+\frac{1}{2}\mathrm{i} & \frac{1}{2} & 1+\frac{1}{2}\mathrm{i} \\ -\frac{1}{2}-\frac{1}{2}\mathrm{i} & 1-\frac{1}{2}\mathrm{i} & \frac{73}{2} \end{pmatrix}$$

is also a positive definite matrix. So $\mathbf{H}_2(\mathscr{S})$ is a positive definite matrix. An application of Theorem 9.1 shows that F is a Hurwitz matrix polynomial.

Now we can uncover the connection between “matrix Hurwitz type polynomial” and “Hurwitz matrix polynomial”. This connection bridges our study in Chapter 8 and Chapter 9.

Theorem 9.7. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}_{\infty}^{p\times p}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic with the $(n-1)$-th SRMP $\mathscr{S}_{\langle n-1\rangle}$. The following statements are equivalent:*

(i) *F is a Hurwitz matrix polynomial and $\mathscr{S} \in \mathbb{C}_{\infty,n-1,H}^{p\times p}$.*

(ii) *F is a right matrix Hurwitz type polynomial.*

(iii) *$\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}_{p,n-1}^{>}$.*

In this case, both following statements hold:

(iv) *F is the unique Hurwitz matrix polynomial with the $(n-1)$-th SRMP $\mathscr{S}_{\langle n-1\rangle}$.*

(v) *F is the unique right matrix Hurwitz type polynomial with the $(n-1)$-th SRMP $\mathscr{S}_{\langle n-1\rangle}$.*

Proof. Use Lemma 8.9 and Theorem 9.1. □

Theorem 9.8. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}_{\infty}^{p\times p}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic with the $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1\rangle}$. The following statements are equivalent:*

(i) *F is a Hurwitz matrix polynomial and $\mathscr{S} \in \mathbb{C}_{\infty,n-1,H}^{p\times p}$.*

(ii) *F is a left matrix Hurwitz type polynomial.*

(iii) *$\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}_{p,n-1}^{>}$.*

In this case, both following statements hold:

(iv) *F is the unique Hurwitz matrix polynomial with the $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1\rangle}$.*

(v) *F is the unique left matrix Hurwitz type polynomial with the $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1\rangle}$.*

Proof. Apply Lemma 8.10 and Theorem 9.2. □

As a immediate consequence of Theorems 9.7 and 8.11, a three-term recurrence relation for Hurwitz matrix polynomials is established in terms of their right Hurwitz parametrizations.

Theorem 9.9. *Let $n \in \mathbb{N} \cup \infty$. Let $\{\mathbf{c}_k\}_{k=0}^{[\frac{n}{2}]} \cup \{\mathbf{d}_k\}_{k=0}^{[\frac{n-1}{2}]} \subseteq \mathbb{C}_{>}^{p\times p}$ and let $(F_l)_{l=1}^{n}$ be defined as in (8.1)–(8.3). Then for each $l \in \mathbb{Z}_{1,n}$, F_l is the unique Hurwitz matrix polynomial with the right Hurwitz parametrization $[(\mathbf{c}_j)_{j=0}^{[\frac{l}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{l-1}{2}]}]$.*

Analogously, by Theorems 9.8 and 8.12 we obtain a three-term recurrence relation for Hurwitz matrix polynomials in terms of their left Hurwitz parametrizations.

Theorem 9.10. *Let $n \in \mathbb{N} \cup \infty$. Let $\{\mathbf{c}_k\}_{k=0}^{[\frac{n}{2}]} \cup \{\mathbf{d}_k\}_{k=0}^{[\frac{n-1}{2}]} \subseteq \mathbb{C}_{>}^{p\times p}$ and let $(F_l)_{l=1}^{n}$ be defined as in (8.14)–(8.16). Then for each $l \in \mathbb{Z}_{1,n}$, F_l is the unique Hurwitz matrix polynomial with the left Hurwitz parametrization $[(\mathbf{c}_j)_{j=0}^{[\frac{l}{2}]}, (\mathbf{d}_j)_{j=0}^{[\frac{l-1}{2}]}]$.*

9.2 $\mathscr{S}$-system of Hurwitz matrix polynomials

In the previous section, we established a bijective correspondence between Hurwitz matrix polynomials and Stieltjes positive definite sequences. Accordingly, starting from a matrix sequence $\mathscr{S}$ with $\mathscr{S}_{\langle n-1\rangle}$ being Stieltjes positive definite, we introduce a particular sequence of Hurwitz matrix polynomials $(F_k)_{k=1}^{n}$ of order n relative to $\mathscr{S}$ in this section.

It was also realized in the previous section that the two notions "Hurwitz matrix polynomials" and "matrix Hurwitz type polynomials" are equivalent. Hence, a certain three-terms recurrence relation of the sequence $(F_k)_{k=1}^{n}$ follows from the appropriate relation of right matrix Hurwitz type polynomials constructed in Chapter 8, where the related matrix coefficients are derived from the DS-parametrization of $\mathscr{S}_{\langle n-1\rangle}$. We also show the above mentioned matrix coefficients are connected to the Favard pair due to the relations between the DS-parametrization of $\mathscr{S}_{\langle n-1\rangle}$ and the Favard pair (see [18]). In addition, this sequence $(F_k)_{k=1}^n$ involves the interesting property that a special matrix rational function formed by F_k and F_{k+1} admits a certain finite matrix continued fraction of Jacobi type and the important Christoffel-Darboux relations hold for $(F_k)_{k=1}^n$.

Based on the recent investigations of Hankel positive definitive sequences and Stieltjes positive definitive sequences (see e.g. [28, 31]), we further consider the one-step extension problem for this class of Hurwitz matrix polynomials $(F_k)_{k=1}^n$.

Definition 9.11. Let $n \in \mathbb{N}\cup\infty$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$ be such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{>}_{p,n-1}$. $(F_k)_{k=1}^n$ is called a *left* (resp. *right*) *$\mathscr{S}$-system of Hurwitz matrix polynomials of order* n if for each $k \in \mathbb{Z}_{1,n}$, $F_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$ is a Hurwitz matrix polynomial with the $(k-1)$-th SLMP (resp. SRMP) $\mathscr{S}_{\langle k-1\rangle}$.

Remark 9.12. Let $n \in \mathbb{N}\cup\infty$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$ be such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{>}_{p,n-1}$. Further for each $k \in \mathbb{Z}_{1,n}$, let $F_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$. Then $(F_k)_{k=1}^n$ is a right $\mathscr{S}$-system of Hurwitz matrix polynomials of order n if and only if $(F_k^{\vee})_{k=1}^n$ is a left $\mathscr{S}$-system of Hurwitz matrix polynomials of order n.

Let $n \in \mathbb{N}$ and let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be a Hurwitz matrix polynomial. By Theorem 9.7 one can see that F is a right matrix Hurwitz type polynomial. Let $\mathscr{S}_{n-1}$ be $(n-1)$-th SRMP of F. Let $m \in \mathbb{N}$ be chosen such that $n = 2m$ or $n = 2m+1$ is satisfied. By Theorem 7.12 of [17] we have:

(i) Suppose that $n = 2m$ and $[(\mathbf{L}_k)_{k=0}^{m-1}, (\mathbf{M}_k)_{k=0}^{m-1}]$ is the DS-parametrization of $\mathscr{S}_{n-1}$. Then $[(\mathbf{M}_k)_{k=0}^{m-1}, (\mathbf{L}_k)_{k=0}^{m-1}]$ is the unique right Hurwitz parametrization of F.

(ii) Suppose that $n = 2m-1$ and $[(\mathbf{L}_k)_{k=0}^{m-2}, (\mathbf{M}_k)_{k=0}^{m-1}]$ is the DS-parametrization of $\mathscr{S}_{n-1}$. Then $[(\mathbf{M}_k)_{k=0}^{m-1}, (\mathbf{L}_k)_{k=0}^{m-2}]$ is the unique right Hurwitz parametrization of F.

In view of Theorem 9.10, we can immediately build the unique three-term recurrence relation for the left $\mathscr{S}$-system of Hurwitz matrix polynomials of order n in terms of $\mathscr{S}$.

Theorem 9.13. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$ be such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{>}_{p,n-1}$. Let $[(\mathbf{L}_k)_{k=0}^{[\frac{n-1}{2}]}, (\mathbf{M}_k)_{k=0}^{[\frac{n}{2}]}]$ be the DS-parametrization of $\mathscr{S}_{\langle n-1\rangle}$ and let $(F_l)_{l=0}^n$ be given via the following recurrence form:*

$$F_{2k}(z) = zF_{2k-1}(z) + F_{2k-2}(z)\mathbf{A}_k, \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]} \tag{9.2}$$

$$F_{2k+1}(z) = zF_{2k}(z) + F_{2k-1}(z)\mathbf{B}_k, \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]} \tag{9.3}$$

with the initial values

$$F_0(z) = I_p, \quad F_1(z) = zI_p + \mathbf{M}_0^{-1}, \tag{9.4}$$

where

$$\mathbf{A}_k := \mathbf{N}_{k-2}\mathbf{N}_{k-1}^{-1}, \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]}, \tag{9.5}$$

$$\mathbf{B}_k := \mathbf{N}_{k-2}\mathbf{M}_{k-1}\mathbf{M}_k^{-1}\mathbf{N}_{k-1}^{-1}, \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}, \tag{9.6}$$

$$\mathbf{N}_k := \begin{cases} I_p, & k = -1, \\ \overrightarrow{\prod}_{j=0}^{k}\mathbf{M}_j\mathbf{L}_j, & k \in \mathbb{Z}_{0,[\frac{n}{2}]}. \end{cases} \tag{9.7}$$

Then $(F_k)_{k=1}^n$ is the unique right $\mathscr{S}$-system of Hurwitz matrix polynomials of order n.

Corollary 9.14. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}_{\infty,n-1,H}^{p\times p}$ be such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}_{p,n-1}^{>}$. Let $[(\mathbf{L}_k)_{k=0}^{[\frac{n-1}{2}]}, (\mathbf{M}_k)_{k=0}^{[\frac{n}{2}]}]$ be the DS-parametrization of $\mathscr{S}_{\langle n-1\rangle}$ and let $(F_k)_{k=1}^n$ be the right $\mathscr{S}$-system of Hurwitz matrix polynomials of order n. Then*

$$F_{2k}(0) = \overleftarrow{\prod}_{j=0}^{k-1}\mathbf{L}_j^{-1}\mathbf{M}_j^{-1}, \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]},$$

$$F_{2k+1}(0) = \mathbf{M}_k^{-1}\overleftarrow{\prod}_{j=0}^{k-1}\mathbf{L}_j^{-1}\mathbf{M}_j^{-1}, \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}.$$

Proof. In the case that $z = 0$, (9.2) and (9.3) turn out that

$$F_{2k}(0) = F_{2k-2}(0)\mathbf{A}_k, \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]}, \tag{9.8}$$

$$F_{2k+1}(0) = F_{2k-1}(0)\mathbf{B}_k, \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}. \tag{9.9}$$

By iterating (9.8) for $k \in \mathbb{Z}_{1,[\frac{n}{2}]}$, we get that

$$F_{2k}(0) = F_0(0)\overrightarrow{\prod}_{j=0}^{k}\mathbf{A}_j = \mathbf{N}_{-1}\mathbf{N}_{k-1}^{-1} = \overleftarrow{\prod}_{j=0}^{k-1}\mathbf{L}_j^{-1}\mathbf{M}_j^{-1}, \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]}.$$

Analogously, the iteration of (9.9) for $k \in \mathbb{Z}_{1,[\frac{n}{2}]}$ shows that

$$\begin{aligned} F_{2k+1}(0) =& F_1(0)\overrightarrow{\prod}_{j=0}^{k}\mathbf{B}_j = \mathbf{M}_0^{-1}\mathbf{N}_{-1}\mathbf{M}_0\mathbf{M}_k^{-1}\mathbf{N}_{k-1}^{-1} \\ =& \mathbf{M}_k^{-1}\mathbf{N}_{k-1}^{-1} = \mathbf{M}_k^{-1}\overleftarrow{\prod}_{j=0}^{k-1}\mathbf{L}_j^{-1}\mathbf{M}_j^{-1}, \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}. \end{aligned}$$

□

As an immediate consequence of Remark 9.12 and Theorem 9.13, we get the dual consequence for the right $\mathscr{S}$-system of Hurwitz matrix polynomials.

Theorem 9.15. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}_{\infty,n-1,H}^{p\times p}$ be such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}_{p,n-1}^{>}$. Let $[(\mathbf{L}_k)_{k=0}^{[\frac{n-1}{2}]}, (\mathbf{M}_k)_{k=0}^{[\frac{n}{2}]}]$ be the DS-parametrization of $\mathscr{S}_{\langle n-1\rangle}$ and let $(F_l)_{l=1}^n$ be given via the following recurrence form:*

$$F_{2k}(z) = zF_{2k-1}(z) + \mathbf{A}_k^*F_{2k-2}(z), \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]}, \tag{9.10}$$

$$F_{2k+1}(z) = zF_{2k}(z) + \mathbf{B}_k^*F_{2k-1}(z), \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}, \tag{9.11}$$

with the initial values

$$F_0(z) = I_p, \quad F_1(z) = zI_p + \mathbf{M}_0^{-1}, \tag{9.12}$$

where $(\mathbf{A}_k)_{k=1}^{[\frac{n}{2}]}$ *and* $(\mathbf{B}_k)_{k=1}^{[\frac{n-1}{2}]}$ *are given via the formulas* (9.5)–(9.7). *Then* $(F_l)_{l=1}^{n}$ *is the unique left* $\mathscr{S}$*-system of Hurwitz matrix polynomials of order* n.

In view of Remark 9.12 and Corollary 9.14, we have

Corollary 9.16. *Let* $n \in \mathbb{N}$ *and let* $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$ *be such that* $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{>}_{p,n-1}$. *Let* $[(\mathbf{L}_k)_{k=0}^{[\frac{n-1}{2}]}, (\mathbf{M}_k)_{k=0}^{[\frac{n}{2}]}]$ *be the DS-parametrization of* $\mathscr{S}_{\langle n-1\rangle}$ *and let* $(F_k)_{k=1}^{n}$ *be the left* $\mathscr{S}$*-system of Hurwitz matrix polynomials of order* n. *Then*

$$\begin{aligned} F_{2k}(0) &= \overrightarrow{\prod}_{j=0}^{k-1}\mathbf{M}_j^{-1}\mathbf{L}_j^{-1}, \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]}, \\ F_{2k+1}(0) &= \left(\overrightarrow{\prod}_{j=0}^{k-1}\mathbf{M}_j^{-1}\mathbf{L}_j^{-1}\right)\mathbf{M}_k^{-1}, \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}. \end{aligned}$$

From Theorem 9.13 (resp. Theorem 9.15) we infer that the ordered pair $[(\mathbf{A}_k)_{k=1}^{[\frac{n}{2}]}, (\mathbf{B}_k)_{k=1}^{[\frac{n-1}{2}]}]$, uniquely determined by $\mathscr{S}_{\langle n-1\rangle}$, forms the coefficients of the three-terms relation of the right (resp. left) $\mathscr{S}$-system of Hurwitz matrix polynomials of order n. Then we call that $[(\mathbf{A}_k)_{k=1}^{[\frac{n}{2}]}, (\mathbf{B}_k)_{k=1}^{[\frac{n-1}{2}]}]$ is the *Hurwitz pair associated with* $\mathscr{S}_{\langle n-1\rangle}$.

Next we will introduce the Favard pair associated with a finite Stieltjes positive definite sequence, which appears in the study of MLOSMP or MROSMP (see [18, Sections 8-9]). This terminology will prove to be relevant to the Hurwitz pair.

Definition 9.17. Let $n \in \mathbb{Z}_{3,\infty}$ and let $\mathscr{S}_{n-1} := (s_j)_{j=0}^{n-1} \in \mathcal{K}^{>}_{p,n-1}$. Let

$$\mathscr{A}_k := \mathbf{E}_k \cdot \mathbf{L}_{1,k}(\mathscr{S}), \quad k \in \mathbb{Z}_{0,[\frac{n}{2}]-1}$$

and

$$\mathscr{B}_k := \mathbf{L}_{1,k}^{-1}(\mathscr{S}) \cdot \mathbf{L}_{1,k+1}(\mathscr{S}), \quad k \in \mathbb{Z}_{0,[\frac{n-1}{2}]-1},$$

where

$$\mathbf{E}_k := \begin{cases} s_1, & \text{if } k = 0, \\ (-\mathbf{Z}_{k,2k-1}(\mathscr{S})\mathbf{L}_{1,k-1}^{-1}(\mathscr{S}), I_p)\mathbf{H}_k(\Delta\mathscr{S}) \cdot \begin{pmatrix} -\mathbf{L}_{1,k-1}^{-1}(\mathscr{S})\mathbf{Y}_{k,2k-1}(\mathscr{S}) \\ I_p \end{pmatrix}, & \text{if } k \in \mathbb{Z}_{0,[\frac{n}{2}]-1}. \end{cases}$$

Then the ordered pair $[(\mathscr{A}_k)_{k=0}^{[\frac{n}{2}]-1}, (\mathscr{B}_k)_{k=0}^{[\frac{n-1}{2}]-1}]$ is called the *Favard pair associated with* $\mathscr{S}_{n-1}$.

Next we will show the relevance between the Favard pairs and the Hurwitz pairs associated with Stieltjes positive definite sequences.

Proposition 9.18. *Let $n \in \mathbb{Z}_{3,\infty}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$ be such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{>}_{p,n-1}$. Let $[(\mathbf{A}_k)_{k=1}^{[\frac{n}{2}]}, (\mathbf{B}_k)_{k=1}^{[\frac{n-1}{2}]}]$ be the Hurwitz pair associated with $\mathscr{S}_{\langle n-1\rangle}$ and let $[(\mathscr{A}_{1,k})_{k=0}^{[\frac{n}{2}]-1}, (\mathscr{B}_{1,k})_{k=0}^{[\frac{n-1}{2}]-1}]$ and $[(\mathscr{A}_{2,k})_{k=0}^{[\frac{n-1}{2}]-1}, (\mathscr{B}_{2,k})_{k=0}^{[\frac{n}{2}]-2}]$ be the Favard pair associated with $\mathscr{S}_{\langle n-1\rangle}$ and $(\Delta\mathscr{S})_{\langle n-2\rangle}$, respectively. Then*

$$
\begin{aligned}
\mathscr{A}_{1,k+1} &:= \mathbf{A}^*_{k+2} + \mathbf{B}^*_{k+1}, \quad k \in \mathbb{Z}_{0,[\frac{n}{2}]-2},\\
\mathscr{B}_{1,k} &:= \mathbf{A}_{k+1}\mathbf{B}_{k+1}, \quad k \in \mathbb{Z}_{0,[\frac{n-1}{2}]-1},\\
\mathscr{A}_{2,k} &:= \mathbf{A}^*_{k+1} + \mathbf{B}^*_{k+1}, \quad k \in \mathbb{Z}_{0,[\frac{n-1}{2}]-1},\\
\mathscr{B}_{2,k} &:= \mathbf{B}_{k+1}\mathbf{A}_{k+2}, \quad k \in \mathbb{Z}_{0,[\frac{n}{2}]-2}.
\end{aligned}
$$

Proof. Suppose that $[(\mathbf{L}_k)_{k=0}^{[\frac{n-1}{2}]}, (\mathbf{M}_k)_{k=0}^{[\frac{n}{2}]}]$ is the DS-parametrization of $\mathscr{S}_{\langle n-1\rangle}$ and $\mathbf{N}_k$ is given as in (9.7). In view of [18, Theorem 9.3], one can see for each $k \in \mathbb{Z}_{0,[\frac{n}{2}]-2}$

$$
\begin{aligned}
\mathscr{A}_{1,k+1} =& \left[\overrightarrow{\prod}_{j=0}^{k+1}\left(\mathbf{M}_j^{-1}\mathbf{L}_j^{-1}\right)\right](\mathbf{L}_k + \mathbf{L}_{k+1})\mathbf{M}_k\left[\overleftarrow{\prod}_{j=0}^{k-1}(\mathbf{L}_j\mathbf{M}_j)\right]\\
=&\left[\overrightarrow{\prod}_{j=0}^{k+1}\left(\mathbf{M}_j^{-1}\mathbf{L}_j^{-1}\right)\right]\left[\overleftarrow{\prod}_{j=0}^{k}(\mathbf{L}_j\mathbf{M}_j)\right] + \left[\overrightarrow{\prod}_{j=0}^{k}\left(\mathbf{M}_j^{-1}\mathbf{L}_j^{-1}\right)\right]\mathbf{M}_{k+1}^{-1}\mathbf{M}_k\cdot\\
&\left[\overleftarrow{\prod}_{j=0}^{k-1}(\mathbf{L}_j\mathbf{M}_j)\right]\\
=&\mathbf{N}_{k+1}^{-*}\mathbf{N}_k^* + \mathbf{N}_k^{-*}\mathbf{M}_{k+1}^{-*}\mathbf{M}_k^*\mathbf{N}_{k-1}^*\\
=&\mathbf{A}^*_{k+2} + \mathbf{B}^*_{k+1},
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{B}_{2,k} =& \left[\overrightarrow{\prod}_{j=0}^{k-1}(\mathbf{M}_j\mathbf{L}_j)\right]\mathbf{M}_k\mathbf{M}_{k+1}^{-1}\left[\overleftarrow{\prod}_{j=0}^{k+1}\left(\mathbf{L}_j^{-1}\mathbf{M}_j^{-1}\right)\right]\\
=&\mathbf{N}_{k-1}\mathbf{M}_k\mathbf{M}_{k+1}^{-1}\mathbf{N}_{k+1}^{-1}\\
=&\mathbf{N}_{k-1}\mathbf{M}_k\mathbf{M}_{k+1}^{-1}\mathbf{N}_k^{-1}\mathbf{N}_k\mathbf{N}_{k+1}^{-1}\\
=&\mathbf{B}_{k+1}\mathbf{A}_{k+2}
\end{aligned}
$$

and for each $k \in \mathbb{Z}_{0,[\frac{n-1}{2}]-1}$,

$$\begin{aligned}
\mathscr{A}_{2,k} &= \left[\overrightarrow{\prod}_{j=0}^{k}\left(\mathbf{M}_j^{-1}\mathbf{L}_j^{-1}\right)\right]\mathbf{M}_{k+1}^{-1}(\mathbf{M}_k+\mathbf{M}_{k+1})\left[\overleftarrow{\prod}_{j=0}^{k-1}(\mathbf{L}_j\mathbf{M}_j)\right]\\
&= \left[\overrightarrow{\prod}_{j=0}^{k}\left(\mathbf{M}_j^{-1}\mathbf{L}_j^{-1}\right)\right]\left[\overleftarrow{\prod}_{j=0}^{k-1}(\mathbf{L}_j\mathbf{M}_j)\right] + \left[\overrightarrow{\prod}_{j=0}^{k}\left(\mathbf{M}_j^{-1}\mathbf{L}_j^{-1}\right)\right]\mathbf{M}_{k+1}^{-1}\mathbf{M}_k\cdot\\
&\quad\left[\overleftarrow{\prod}_{j=0}^{k-1}(\mathbf{L}_j\mathbf{M}_j)\right]\\
&=\mathbf{N}_k^{-*}\mathbf{N}_{k-1}^{*}+\mathbf{N}_k^{-*}\mathbf{M}_{k+1}^{-*}\mathbf{M}_k^{*}\mathbf{N}_{k-1}^{*}\\
&=\mathbf{A}_{k+1}^{*}+\mathbf{B}_{k+1}^{*},
\end{aligned}$$

$$\begin{aligned}
\mathscr{B}_{1,k} &= \left[\overrightarrow{\prod}_{j=0}^{k-1}(\mathbf{M}_j\mathbf{L}_j)\right]\mathbf{L}_k^{-1}\mathbf{M}_{k+1}^{-1}\left[\overleftarrow{\prod}_{j=0}^{k}\left(\mathbf{L}_j^{-1}\mathbf{M}_j^{-1}\right)\right]\\
&=\mathbf{N}_{k-1}\mathbf{L}_k^{-1}\mathbf{M}_{k+1}^{-1}\mathbf{N}_k^{-1}\\
&=\mathbf{N}_{k-1}\mathbf{N}_k^{-1}\mathbf{N}_{k-1}\mathbf{M}_k\mathbf{M}_{k+1}^{-1}\mathbf{N}_k^{-1}\\
&=\mathbf{A}_{k+1}\mathbf{B}_{k+1}.
\end{aligned}$$

□

The following proposition connects the right $\mathscr{S}$-system of Hurwitz matrix polynomials with a certain finite matrix continued fraction of Jacobi type.

Proposition 9.19. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$ be such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{>}_{p,n-1}$. Let $\mathbf{A}_0 := \mathscr{S}_{\langle 0\rangle}$, $\mathbf{B}_0 := I_p$ and let $[(\mathbf{A}_k)_{k=1}^{[\frac{n}{2}]}, (\mathbf{B}_k)_{k=1}^{[\frac{n-1}{2}]}]$ be the Hurwitz pair associated with $\mathscr{S}_{\langle n-1\rangle}$. Further, let $(F_l)_{l=1}^{n}$ be the unique left $\mathscr{S}$-system of Hurwitz matrix polynomials of order n. Then for $k \in \mathbb{Z}_{1,[\frac{n}{2}]}$,*

$$\frac{F_{2k-1}(z)}{F_{2k}(z)} = \cfrac{I_p}{zI_p + \cfrac{\mathbf{A}_k^*}{zI_p + \cfrac{\mathbf{B}_{k-1}^*}{zI_p + \cfrac{\ddots}{zI_p + \cfrac{\mathbf{B}_0^*}{zI_p + \mathbf{A}_0^*}}}}} \tag{9.13}$$

and for $k \in \mathbb{Z}_{0,[\frac{n-1}{2}]}$,

$$\frac{F_{2k}(z)}{F_{2k+1}(z)} = \cfrac{I_p}{zI_p + \cfrac{\mathbf{B}_k^*}{zI_p + \cfrac{\mathbf{A}_k^*}{zI_p + \cfrac{\ddots}{zI_p + \cfrac{\mathbf{B}_0^*}{zI_p + \mathbf{A}_0^*}}}}}. \tag{9.14}$$

Proof. We denote

$$R_l(z) := \frac{F_l(z)}{F_{l+1}(z)}, \quad l \in \mathbb{Z}_{0,n-1}.$$

Rewriting (9.10)–(9.12), we obtain

$$R_{2k-1}(z) = \frac{I_p}{zI_p + \mathbf{A}_k^* R_{2k-2}(z)}, \quad k \in \mathbb{Z}_{1,[\frac{n}{2}]}$$

and

$$R_{2k}(z) = \frac{I_p}{zI_p + \mathbf{B}_k^* R_{2k-1}(z)}, \quad k \in \mathbb{Z}_{1,[\frac{n-1}{2}]}.$$

By recursively applying these formulas, we can represent R_{2k-1} and R_{2k} as the continuous fraction forms (9.13) and (9.14), respectively.

□

Next we will present the Christoffel-Darboux formula for the left $\mathscr{S}$-system of Hurwitz matrix polynomials.

Proposition 9.20. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$ be such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{>}_{p,n-1}$. Let $[(\mathbf{A}_k)_{k=1}^{[\frac{n}{2}]}, (\mathbf{B}_k)_{k=1}^{[\frac{n-1}{2}]}]$ be the Hurwitz pair associated with $\mathscr{S}_{\langle n-1\rangle}$. Further, let $(F_l)_{l=1}^{n}$ be the unique left $\mathscr{S}$-system of Hurwitz matrix polynomials of order n. Then for each $l \in \mathbb{Z}_{0,n-1}$,*

$$(z-\omega)\sum_{j=0}^{l} F_j^{\vee}(z)\mathbf{X}_j F_j(\omega) = F_{l+1}^{\vee}(z)\mathbf{X}_l^* F_l(\omega) - F_l^{\vee}(z)\mathbf{X}_l F_{l+1}(\omega), \tag{9.15}$$

where $\mathbf{X}_0 \in \mathbb{C}^{p\times p}_H$ is such that $\mathbf{X}_0$ commutes with $\mathscr{S}_{\langle 0\rangle}$ and

$$\mathbf{X}_{2k} := \mathbf{X}_0 \left(\overrightarrow{\prod}_{j=1}^{k} \mathbf{A}_j \mathbf{B}_j\right)^{-*}, \quad k \in \mathbb{Z}_{1,\left[\frac{n-1}{2}\right]}, \tag{9.16}$$

$$\mathbf{X}_{2k+1} := -\mathbf{X}_0 \left(\overrightarrow{\prod}_{j=1}^{k} \mathbf{A}_j \mathbf{B}_j\right)^{-*} \mathbf{A}_{k+1}^{-*}, \quad k \in \mathbb{Z}_{0,\left[\frac{n-2}{2}\right]}. \tag{9.17}$$

Proof. The proof is conducted by induction. We have

$$\begin{aligned}(z-\omega)F_0^{\vee}(z)\mathbf{X}_0F_0(\omega) &= (z-\omega)\mathbf{X}_0 = \left(zI_p + \mathscr{S}_{\langle 0\rangle}\right)\mathbf{X}_0 - \mathbf{X}_0\left(\omega I_p + \mathscr{S}_{\langle 0\rangle}\right)\\ &= F_1^{\vee}(z)\mathbf{X}_0^*F_0(\omega) - F_0^{\vee}(z)\mathbf{X}_0F_1(\omega).\end{aligned}$$

In other words, (9.15) holds for $l = 0$.

Assuming that (9.15) holds for $l = 2k$, we will now prove that it holds for $l = 2k+1$. We have

$$
\begin{aligned}
&(z-\omega)\sum_{j=0}^{2k+1} F_j^\vee(z)\mathbf{X}_j F_j(\omega)\\
=&(z-\omega)\sum_{j=0}^{2k} F_j^\vee(z)\mathbf{X}_j F_j(\omega) + (z-\omega)F_{2k+1}^\vee(z)\mathbf{X}_{2k+1}F_{2k+1}(\omega)\\
=&F_{2k+1}^\vee(z)\mathbf{X}_{2k}F_{2k}(\omega) - F_{2k}^\vee(z)\mathbf{X}_{2k}^*F_{2k+1}(\omega) + (z-\omega)F_{2k+1}^\vee(z)\mathbf{X}_{2k+1}F_{2k+1}(\omega)\\
=&F_{2k+1}^\vee(z)\mathbf{X}_{2k+1}\left(\mathbf{X}_{2k+1}^{-1}\mathbf{X}_{2k}F_{2k}(\omega) - \omega F_{2k+1}(\omega)\right) - \left(F_{2k}^\vee(z)\mathbf{X}_{2k}^*\mathbf{X}_{2k+1}^{-*} - zF_{2k+1}^*(z)\right)\\
&\cdot \mathbf{X}_{2k+1}^* F_{2k+1}(\omega)\\
=&-F_{2k+1}^\vee(z)\mathbf{X}_{2k+1}\left(\mathbf{A}_{k+1}^*F_{2k}(\omega) + \omega F_{2k+1}(\omega)\right) + \left(F_{2k}^\vee(z)\mathbf{A}_{k+1} + zF_{2k+1}^*(z)\right)\mathbf{X}_{2k+1}^*\\
&\cdot F_{2k+1}(\omega)\\
=&F_{2k+2}^\vee(z)\mathbf{X}_{2k+1}^*F_{2k+1}(\omega) - F_{2k+1}^\vee(z)\mathbf{X}_{2k+1}F_{2k+2}(\omega).
\end{aligned}
$$

This implies (9.15) for $l = 2k+1$.

Analogously, by assuming that (9.15) holds for $l = 2k-1$, we obtain

$$
\begin{aligned}
&(z-\omega)\sum_{j=0}^{2k} F_j^\vee(z)\mathbf{X}_j F_j(\omega)\\
=&(z-\omega)\sum_{j=0}^{2k-1} F_j^\vee(z)\mathbf{X}_j F_j(\omega) + (z-\omega)F_{2k}^*(z)\mathbf{X}_{2k}F_{2k}(\omega)\\
=&F_{2k}^\vee(z)\mathbf{X}_{2k-1}F_{2k-1}(\omega) - F_{2k-1}^\vee(z)\mathbf{X}_{2k-1}^*F_{2k}(\omega) + (z-\omega)F_{2k}^\vee(z)\mathbf{X}_{2k}F_{2k}(\omega)\\
=&F_{2k}^\vee(z)\mathbf{X}_{2k}\left(\mathbf{X}_{2k}^{-1}\mathbf{X}_{2k-1}F_{2k-1}(\omega) - \omega F_{2k}(\omega)\right) - \left(F_{2k-1}^\vee(z)\mathbf{X}_{2k-1}^*\mathbf{X}_{2k}^{-*} - zF_{2k}^\vee(z)\right)\\
&\cdot \mathbf{X}_{2k}^* F_{2k}(\omega)\\
=&-F_{2k}^\vee(z)\mathbf{X}_{2k-1}\left(\mathbf{B}_k^*F_{2k-1}(\omega) + \omega F_{2k}(\omega)\right) + \left(F_{2k-1}^\vee(z)\mathbf{B}_k + zF_{2k}^\vee(z)\right)\mathbf{X}_{2k}^*\\
&\cdot F_{2k}(\omega)\\
=&F_{2k+1}^\vee(z)\mathbf{X}_{2k}^*F_{2k}(\omega) - F_{2k}^\vee(z)\mathbf{X}_{2k}F_{2k+1}(\omega),
\end{aligned}
$$

which implies (9.15) for $l = 2k$. Hence (9.15) holds for each $l \in \mathbb{Z}_{0,n-1}$.

□

As an immediate consequence of Remark 9.12 and Proposition 9.20, we can obtain the Christoffel-Darboux formula for the right $\mathscr{S}$-system of Hurwitz matrix polynomials.

Proposition 9.21. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}_{\infty,n-1,H}^{p\times p}$ be such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}_{p,n-1}^{>}$. Let $(\mathbf{X}_l)_{l=0}^{n}$ be given as in Proposition 9.20. Further, let $(F_l)_{l=1}^{n}$ be the unique right $\mathscr{S}$-system of Hurwitz matrix polynomials of order n. Then for each $l \in \mathbb{Z}_{0,n-1}$,*

$$(z-\omega)\sum_{j=0}^{l} F_j(z)\mathbf{X}_j F_j^\vee(\omega) = F_{l+1}(z)\mathbf{X}_l^* F_l^\vee(\omega) - F_l(z)\mathbf{X}_l F_{l+1}^\vee(\omega). \tag{9.18}$$

Up to now much attention is paid to the description of $\mathscr{S}$-system of Hurwitz matrix polynomials. Next we turn to the extension problem of the $\mathscr{S}$-system of Hurwitz matrix polynomials. We precede our discussion by introducing a corresponding terminology.

Let $n \in \mathbb{N} \cup \infty$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$ be such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{>}_{p,n-1}$. Let $(F_k)_{k=1}^{n}$ be the right (resp. left) $\mathscr{S}$-system of Hurwitz matrix polynomials of order n. Here we call $(F_k)_{k=1}^{n}$ to be *one-step extendable* if there exists a Hurwitz matrix polynomial $F_{n+1} \in \mathcal{P}_{p\times p,n+1,\mathbb{C}}$ such that $(F_k)_{k=1}^{n+1}$ is the right (resp. left) $\mathscr{S}$-system of Hurwitz matrix polynomials of order $n+1$. Since $(F_k)_{k=1}^{n}$ is the right (resp. left) $\mathscr{S}$-system of Hurwitz matrix polynomials of order n, we have $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{>}_{p,n-1}$. Suppose that $\mathscr{S} := (s_j)_{j\in\mathbb{N}_0}$. The key point to meet the requirement that $(F_k)_{k=1}^{n}$ is one-step extendable is to choose a suitable s_n such that $\mathscr{S}_{\langle n\rangle} \in \mathcal{K}^{>}_{p,n}$. In this regard, some auxiliary lemmas are needed.

Lemma 9.22. *Let $m \in \mathbb{N} \cup \infty$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty}$. $\mathscr{S}_{\langle 2m+1\rangle} \in \mathcal{K}^{>}_{p,2m+1}$ if and only if $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{K}^{>}_{p,2m}$ and $\mathbf{L}_{1,m}(\Delta\mathscr{S}) \in \mathbb{C}^{p\times p}_{>}$.*

Proof. The "if" implication: Since $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{K}^{>}_{p,2m}$, we have that $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{H}^{>}_{p,2m}$ and $(\Delta\mathscr{S})_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$. As a result of Proposition 2.24 in [28], the fact $(\Delta\mathscr{S})_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$ and $L_{2,m+1}(\mathscr{S}) \in \mathbb{C}^{p\times p}_{>}$ implies that $(\Delta\mathscr{S})_{\langle 2m\rangle} \in \mathcal{H}^{>}_{p,2m}$. A combination of $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{H}^{>}_{p,2m}$ and $(\Delta\mathscr{S})_{\langle 2m\rangle} \in \mathcal{H}^{>}_{p,2m}$ means that $\mathscr{S}_{\langle 2m+1\rangle} \in \mathcal{K}^{>}_{p,2m+1}$.

The "only if" implication: Suppose that $\mathscr{S}_{\langle 2m+1\rangle} \in \mathcal{K}^{>}_{p,2m+1}$. Then $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{H}^{>}_{p,2m}$ and $(\Delta\mathscr{S})_{\langle 2m\rangle} \in \mathcal{H}^{>}_{p,2m}$. Owing to Proposition 2.24 in [28] again, it follows that $(\Delta\mathscr{S})_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$ and $L_{2,m+1}(\mathscr{S}) \in \mathbb{C}^{p\times p}_{>}$. A combination of $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{H}^{>}_{p,2m}$ and $(\Delta\mathscr{S})_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$ means that $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{K}^{>}_{p,2m}$.

□

Lemma 9.23. *Let $m \in \mathbb{N} \cup \infty$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty}$. $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{K}^{>}_{p,2m}$ if and only if $\mathscr{S}_{\langle 2m-1\rangle} \in \mathcal{K}^{>}_{p,2m-1}$ and $\mathbf{L}_{1,m}(\mathscr{S}) \in \mathbb{C}^{p\times p}_{>}$.*

Proof. The "if" implication: Since $\mathscr{S}_{\langle 2m-1\rangle} \in \mathcal{K}^{>}_{p,2m-1}$, we have that $\mathscr{S}_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$ and $(\Delta\mathscr{S})_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$. As a result of Proposition 2.24 in [28], the fact $\mathscr{S}_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$ and $\mathbf{L}_{1,m}(\mathscr{S}) \in \mathbb{C}^{p\times p}_{>}$ implies that $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{H}^{>}_{p,2m}$. A combination of $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{H}^{>}_{p,2m}$ and $(\Delta\mathscr{S})_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$ means that $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{K}^{>}_{p,2m}$.

The "only if" implication: Suppose that $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{K}^{>}_{p,2m}$. Then $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{H}^{>}_{p,2m}$ and $(\Delta\mathscr{S})_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$. Owing to Proposition 2.24 in [28] again, it follows that $\mathscr{S}_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$ and $\mathbf{L}_{1,m}(\mathscr{S}) \in \mathbb{C}^{p\times p}_{>}$. A combination of $\mathscr{S}_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$ and $(\Delta\mathscr{S})_{\langle 2m-2\rangle} \in \mathcal{H}^{>}_{p,2m-2}$ means that $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{K}^{>}_{p,2m}$.

□

In view of Lemma 9.22, we easily obtain a sufficient and necessary condition for the $\mathscr{S}$-system of Hurwitz matrix polynomials of odd order being one-step extendable.

Theorem 9.24. *Let $m \in \mathbb{N} \cup \infty$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,2m,H}$ be such that $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{K}^{>}_{p,2m}$. Let $(F_k)_{k=1}^{2m+1}$ be the right (resp. left) $\mathscr{S}$-system of Hurwitz matrix polynomials of order $2m+1$. $(F_k)_{k=1}^{2m+1}$ is one-step extendable if and only if $\mathbf{L}_{1,m}(\Delta\mathscr{S}) \in \mathbb{C}^{p\times p}_{>}$.*

Analogously, by using Lemma 9.23, we have the dual result for the $\mathscr{S}$-system of Hurwitz matrix polynomials of even order.

Theorem 9.25. *Let $m \in \mathbb{N} \cup \infty$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,2m-1,H}$ be such that $\mathscr{S}_{\langle 2m-1\rangle} \in \mathcal{K}^{>}_{p,2m-1}$. Let $(F_k)_{k=1}^{2m}$ be the right (resp. left) $\mathscr{S}$-system of Hurwitz matrix polynomials of order $2m$. $(F_k)_{k=1}^{2m}$ is one-step extendable if and only if $\mathbf{L}_{1,m}(\mathscr{S}) \in \mathbb{C}^{p\times p}_{>}$.*

In the following, we recall the Stieltjes quadruple of sequences of left orthogonal matrix polynomials (or short SQSLOMP) associated with a Stieltjes positive definite sequence, which plays an important role for the description of the Stieltjes transform of the solution sets of Stieltjes matrix moment problems.

Let, for each $j \in \mathbb{N}$,

$$T_0 := 0_p, \quad T_j := \begin{pmatrix} 0_{p\times jp} & 0_p \\ I_{jp} & 0_{jp\times p} \end{pmatrix}.$$

Obviously, $I_{(j+1)p} - zT_j$ is nonsingular. Let the function $R_j : \mathbb{C} \to \mathbb{C}^{(j+1)p\times(j+1)p}$ be given by

$$R_j(z) := (I_{(j+1)p} - zT_j)^{-1}. \tag{9.19}$$

By a straightforward calculation, it follows that

$$R_j(z) := \begin{pmatrix} I_p & 0_p & 0_p & \cdots & 0_p & 0_p \\ zI_p & I_p & 0_p & \cdots & 0_p & 0_p \\ z^2I_p & zI_p & I_p & \cdots & 0_p & 0_p \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z^jI_p & z^{j-1}I_p & z^{j-2}I_p & \cdots & zI_p & I_p \end{pmatrix}.$$

Definition 9.26. Let $\mathscr{S} \in \mathcal{K}^{>}_{p,\infty}$. Let, for $z \in \mathbb{C}$,

$$P_{1,0}(z) := I_p, \quad Q_{1,0}(z) := 0_p, \quad P_{2,0}(z) := I_p, \quad Q_{2,0}(z) := s_0.$$

For $j \in \mathbb{N}$ and $z \in \mathbb{C}$, let

$$P_{1,j}(z) := (-\mathbf{Z}_{j,2j-1}(\mathscr{S})\,(\mathbf{H}_{j-1}(\mathscr{S}))^{-1}, I_p)R_j(z)\begin{pmatrix} I_p \\ 0_{jp\times p}\end{pmatrix},$$

$$Q_{1,j}(z) := (-\mathbf{Z}_{j,2j-1}(\mathscr{S})\,(\mathbf{H}_{j-1}(\mathscr{S}))^{-1}, I_p)R_j(z)\begin{pmatrix} 0_p \\ \mathbf{Y}_{0,j-1}(\mathscr{S})\end{pmatrix},$$

$$P_{2,j}(z) := (-\mathbf{Z}_{j+1,2j}(\mathscr{S})\left(\mathbf{H}^{\langle 1\rangle}_{j-1}(\mathscr{S})\right)^{-1}, I_p)R_j(z)\begin{pmatrix} I_p \\ 0_{jp\times p}\end{pmatrix},$$

$$P_{2,j}(z) := (-\mathbf{Z}_{j+1,2j}(\mathscr{S})\left(\mathbf{H}^{\langle 1\rangle}_{j-1}(\mathscr{S})\right)^{-1}, I_p)R_j(z)\mathbf{Y}_{0,j-1}(\mathscr{S}).$$

Then $[(P_{1,j})_{j=0}^{\infty}, (Q_{1,j})_{j=0}^{\infty}, (P_{2,j})_{j=0}^{\infty}, (Q_{2,j})_{j=0}^{\infty}]$ is called the *Stieltjes quadruple of sequences of left orthogonal matrix polynomials* (or short SQSLOMP) *associated with* $\mathscr{S}$.

Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{n} \in \mathcal{K}_{p,n}^{>}$. Then there exists a matrix sequence $(s_j)_{j=n+1}^{\infty}$ such that $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{p,\infty}^{>}$. Let $[(P_{1,j})_{j=0}^{\infty}, (Q_{1,j})_{j=0}^{\infty}, (P_{2,j})_{j=0}^{\infty}, (Q_{2,j})_{j=0}^{\infty}]$ be the SQSLOMP associated with $(s_j)_{j=0}^{\infty}$. In view of Definition 9.26, one can see that the quadruple of SQSLOMP $[(P_{1,j})_{j=0}^{[\frac{n+1}{2}]}, (Q_{1,j})_{j=0}^{[\frac{n+1}{2}]}, (P_{2,j})_{j=0}^{[\frac{n}{2}]}, (Q_{2,j})_{j=0}^{[\frac{n}{2}]}]$ depends on the sequence $(s_j)_{j=0}^{n}$. Then $[(P_{1,j})_{j=0}^{[\frac{n+1}{2}]}, (Q_{1,j})_{j=0}^{[\frac{n+1}{2}]}, (P_{2,j})_{j=0}^{[\frac{n}{2}]}, (Q_{2,j})_{j=0}^{[\frac{n}{2}]}]$ is called the *SQSLOMP associated with* $(s_j)_{j=0}^{n}$.

The following theorem establishes a bijective correspondence between the unique left $\mathscr{S}$-system of Hurwitz matrix polynomials of order n and the SQSLOMP associated with $\mathscr{S}_{\langle n\rangle}$.

Theorem 9.27. *Let $n \in \mathbb{N} \cup \infty$ and let $\mathscr{S} \in \mathbb{C}_{\infty,n-1,H}^{p\times p}$ such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}_{p,n-1}^{>}$. Let $F_k \in \mathcal{P}_{p\times p,k,\mathbb{C}}$ be monic for $k \in \mathbb{Z}_{1,n}$. Let $P_0(z) = I_p$, $Q_0(z) = 0_p$ and, for each $k \in \mathbb{Z}_{1,n}$, let*

$$P_k(z) := (-1)^{[\frac{k}{2}]}(F_k)_{\langle e\rangle}(-z) \in \mathcal{P}_{p\times p,[\frac{k}{2}],\mathbb{C}},$$
$$Q_k(z) := (-1)^{[\frac{k-1}{2}]}(F_k)_{\langle o\rangle}(-z) \in \mathcal{P}_{p\times p,[\frac{k-1}{2}],\mathbb{C}}.$$

Then $(F_k)_{k=1}^{n}$ is the unique left $\mathscr{S}$-system of Hurwitz matrix polynomials of order n if and only if $[(P_{2k})_{k=0}^{[\frac{n}{2}]}, (Q_{2k})_{k=0}^{[\frac{n}{2}]}, (Q_{2k-1})_{k=1}^{[\frac{n+1}{2}]}, (P_{2k-1})_{k=1}^{[\frac{n+1}{2}]}]$ is the SQSLOMP associated with $\mathscr{S}_{\langle n-1\rangle}$.

Proof. Remark 7.14 shows that $\mathscr{S}_{\langle k-1\rangle}$ is the k-th SLMP (resp. SRMP) of F_k for $k \in \mathbb{Z}_{1,n}$ if and only if all of the following conditions are satisfied:

(i) $(P_{2j})_{j=0}^{[\frac{n}{2}]}$ is the unique MLOSMP (resp. MROSMP) of order $[\frac{n}{2}]$ with respect to $\mathscr{S}$.

(ii) $(Q_{2j})_{j=0}^{[\frac{n}{2}]}$ is the unique LSMPSK (resp. RSMPSK) of order $[\frac{n}{2}]$ with respect to $\mathscr{S}$.

(iii) $(Q_{2j-1})_{j=1}^{[\frac{n+1}{2}]}$ is the unique MLOSMP (resp. MROSMP) of order $[\frac{n-1}{2}]$ with respect to $\Delta\mathscr{S}$.

(iv) $(P_{2j-1} - Q_{2j-1}s_0)_{j=1}^{[\frac{n+1}{2}]}$ is the unique LSMPSK (resp. RSMPSK) of order $[\frac{n-1}{2}]$ with respect to $\Delta\mathscr{S}$.

Then by using Proposition 4.2 of [18], we complete the proof. □

We conclude this chapter with a new three-term recurrence for the SQSLOMP, which is different from that in Proposition 9.1 of [18].

Theorem 9.28. *Let $n \in \mathbb{N}$ and let $\mathscr{S}_{n-1} \in \mathcal{K}^{>}_{p,n-1}$. Let $[(\mathbf{L}_k)_{k=0}^{[\frac{n-1}{2}]}, (\mathbf{M}_k)_{k=0}^{[\frac{n}{2}]}]$ be the DS-parametrization of $\mathscr{S}_{n-1}$ and let $(P_{2k})_{k=0}^{[\frac{n}{2}]}$, $(Q_{2k})_{k=0}^{[\frac{n}{2}]}$, $(Q_{2k+1})_{k=0}^{[\frac{n-1}{2}]}$, $(P_{2k+1})_{k=0}^{[\frac{n-1}{2}]}$ admit the following recurrence:*

$$P_{2k}(z) = P_{2k-1}(z) - \mathbf{A}_k^* P_{2k-2}(z), \tag{9.20}$$
$$P_{2k+1}(z) = zP_{2k}(z) - \mathbf{B}_k^* P_{2k-1}(z) \tag{9.21}$$

and

$$Q_{2k}(z) = zQ_{2k-1}(z) - \mathbf{A}_k^* Q_{2k-2}(z), \tag{9.22}$$
$$Q_{2k+1}(z) = Q_{2k}(z) - \mathbf{B}_k^* Q_{2k-1}(z) \tag{9.23}$$

with the initial values

$$P_0(z) = 0_p, \quad Q_0(z) = I_p, \quad P_1(z) = \mathbf{M}_0^{-1}, \quad Q_1(z) = I_p, \tag{9.24}$$

where $[(\mathbf{A}_k)_{k=1}^{[\frac{n}{2}]}, (\mathbf{B}_k)_{k=1}^{[\frac{n-1}{2}]}]$ is the Hurwitz pair associated with $\mathscr{S}_{n-1}$. Then $[(P_{2k})_{k=0}^{[\frac{n}{2}]}, (Q_{2k})_{k=0}^{[\frac{n}{2}]}, (Q_{2k+1})_{k=0}^{[\frac{n-1}{2}]}, (P_{2k+1})_{k=0}^{[\frac{n-1}{2}]}]$ is the SQSLOMP associated with $\mathscr{S}_{n-1}$.

Proof. Let $\mathscr{S} \in \mathbb{C}_\infty^{p\times p}$ be such that $\mathscr{S}_{\langle n-1\rangle} := \mathscr{S}_{n-1}$ and let $(F_l)_{l=1}^n$ be the left $\mathscr{S}$-system of Hurwitz matrix polynomials. By using Theorem 9.15, (7.4) and (7.5), we obtain

$$\begin{aligned}(F_{2k})_{\langle e\rangle}(z^2) &= \frac{F_{2k}(z) + F_{2k}(-z)}{2}\\ &= z\frac{F_{2k-1}(z) - F_{2k-1}(-z)}{2} + \mathbf{A}_k^* \frac{F_{2k-2}(z) + F_{2k-2}(-z)}{2}\\ &= z^2 (F_{2k-1})_{\langle o\rangle}(z^2) + \mathbf{A}_k^* (F_{2k-2})_{\langle e\rangle}(z^2), \quad k \in \mathbb{Z}_{0,[\frac{n}{2}]}.\end{aligned} \tag{9.25}$$

Analogously, in view of Theorem 9.15, we rewrite (9.10)–(9.12) in the following form:

$$(F_{2k+1})_{\langle e\rangle}(z^2) = z^2 (F_{2k})_{\langle o\rangle}(z^2) + \mathbf{B}_k^* (F_{2k-1})_{\langle e\rangle}(z^2), \quad k \in \mathbb{Z}_{0,[\frac{n-1}{2}]} \tag{9.26}$$
$$(F_{2k})_{\langle o\rangle}(z^2) = (F_{2k-1})_{\langle e\rangle}(z^2) + \mathbf{A}_k^* (F_{2k-2})_{\langle o\rangle}(z^2), \quad k \in \mathbb{Z}_{0,[\frac{n}{2}]} \tag{9.27}$$
$$(F_{2k+1})_{\langle o\rangle}(z^2) = (F_{2k})_{\langle e\rangle}(z^2) + \mathbf{B}_k^* (F_{2k-1})_{\langle o\rangle}(z^2), \quad k \in \mathbb{Z}_{0,[\frac{n-1}{2}]} \tag{9.28}$$

with the initial values

$$(F_0)_{\langle e\rangle}(z^2) = I_p, \quad (F_0)_{\langle o\rangle}(z^2) = 0_p, \quad (F_1)_{\langle e\rangle}(z^2) = \mathbf{M}_0^{-1}, \quad (F_1)_{\langle o\rangle}(z^2) = I_p. \tag{9.29}$$

Substituting $-z$ for z^2 in the formulas (9.25)–(9.29) and using Theorem 9.27, we can obtain the formulas (9.20)–(9.24).

□

10 Quasi-stable matrix polynomials and some related topics

In this chapter we investigate another important set of matrix polynomials appearing in the theory of stability, called quasi-stable matrix polynomials. Recall that for $n \in \mathbb{N}$ and $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$, F is called a *quasi-stable matrix polynomial* if $\sigma(F) \subseteq \mathbb{C}_- \cup \mathbb{R}$, or equivalently, $\gamma'_+(F) = 0$, $\gamma'_0(F) + \gamma'_-(F) = np$. Obviously, if F is a Hurwitz matrix polynomial, then F is a quasi-stable matrix polynomial.

There are some studies of scalar quasi-stable polynomials in connection to certain Hurwitz matrices. Asner [5] established that the finite Hurwitz matrix of a real quasi-stable polynomial is totally nonnegative. Furthermore, Holtz [44] showed that a polynomial is quasi-stable if and only if its infinite Hurwitz matrix is totally nonnegative (see Theorem 3.44 of [44]). For more literature on the scalar case of these polynomials and their extensions we refer the reader to Dyachenko [21], Kemperman [54].

We continue our strategy from Chapters 7 and 9 to use the Markov parameters approach as the basic instrument. This is due to the fact that concerning quasi-stable matrix polynomials, there are no satisfactory tools generalizing Hurwitz matrices from the scalar case which depend heavily upon the theory of determinants. When one tries to construct the block Hurwitz matrices of which the block elements are the matrix coefficients of a matrix polynomial, it turns out that they do not satisfy several important determinant properties.

Sections 10.1 and 10.2 look for the correspondences between the existence of quasi-stable matrix polynomials with given SLMPs or SRMPs, the solvability of Stieltjes moment problems and the Nevanlinna-Pick interpolation problem for matrix-valued Stieltjes functions. In Section 10.3, we give necessary and sufficient conditions for a monic matrix polynomial F to be a quasi-stable matrix polynomial. In this case, the infinite sequence SRMP (resp. SLMP) of F is described.

10.1 Particular monic quasi-stable matrix polynomials and Stieltjes moment problems

In this section we build particular monic quasi-stable matrix polynomials from Stieltjes nonnegative definite extendable sequences. Our motivation originates from Chapter 7, where we characterized some connection between monic matrix polynomials and monic orthogonal systems of matrix polynomials (see Propositions 7.7–7.12). We aim to construct particular monic quasi-stable matrix polynomials from certain

MLOSMPs or MROSMPs (see Definition 4.21) with respect to Hankel nonnegative definite extendable sequences.

In this connection, we start by setting up specific MLOSMPs with respect to Hankel nonnegative definite extendable sequences, which is a special case of the construction in [28] (see particularly Proposition 3.16, (2.4), (3.8), (3.9) and Lemma 3.14 of [28]).

Let $m \in \mathbb{N}$ and let $\mathscr{S} := (s_j)_{j=0}^{\infty} \in \mathbb{C}_{\infty}^{p\times p}$ such that $\mathscr{S}_{\langle 2m-1\rangle} \in \mathcal{H}_{p,2m-1}^{\geq,e}$. We define the following recurrence relation:

$$\begin{aligned} P_{k+1}^{\diamond}(z) &:= (zI_p - \mathbf{L}_{2,k}(\mathscr{S})(\mathbf{L}_{1,k}(\mathscr{S}))^{\dagger})P_k^{\diamond}(z) \\ &\qquad - \mathbf{L}_{1,k}(\mathscr{S})(\mathbf{L}_{1,k-1}(\mathscr{S}))^{\dagger}P_{k-1}^{\diamond}(z), \end{aligned} \tag{10.1}$$

$$\begin{aligned} Q_{k+1}^{\diamond}(z) &:= (zI_p - \mathbf{L}_{2,k}(\mathscr{S})(\mathbf{L}_{1,k}(\mathscr{S}))^{\dagger})Q_k^{\diamond}(z) \\ &\qquad - \mathbf{L}_{1,k}(\mathscr{S})(\mathbf{L}_{1,k-1}(\mathscr{S}))^{\dagger}Q_{k-1}^{\diamond}(z), \end{aligned} \tag{10.2}$$

for $k \in \mathbb{Z}_{1,m-1}$, with the initial condition:

$$P_0^{\diamond}(z) := I_p,\ P_1^{\diamond}(z) := zI_p - s_1 s_0^{\dagger},\ Q_0^{\diamond}(z) := 0_p,\ Q_1^{\diamond}(z) := s_0, \tag{10.3}$$

where $\mathbf{L}_{2,0}(\mathscr{S}) := s_1$ and for $k \in \mathbb{Z}_{1,m-1}$,

$$\begin{aligned} \mathbf{L}_{2,k}(\mathscr{S}) &:= s_{2k+1} - M_k - \mathbf{L}_{1,k}(\mathscr{S})(\mathbf{L}_{1,k-1}(\mathscr{S}))^{\dagger}(s_{2k-1} - M_{k-1}), \\ M_k &:= \mathbf{Z}_{k,2k-1}(\mathscr{S})(\mathbf{H}_{k-1}(\mathscr{S}))^{\dagger}\mathbf{Y}_{k,2k-1}(\mathscr{S}). \end{aligned}$$

Analogously, let $m \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}_{\infty}^{p\times p}$ such that $(\Delta\mathscr{S})_{\langle 2m-1\rangle} \in \mathcal{H}_{p,2m-1}^{\geq,e}$. We define

$$\begin{aligned} \widetilde{P}_{k+1}^{\diamond}(z) &:= (zI_p - \mathbf{L}_{2,k}(\Delta\mathscr{S})(\mathbf{L}_{1,k}(\Delta\mathscr{S}))^{\dagger})\widetilde{P}_k^{\diamond}(z) \\ &\qquad - \mathbf{L}_{1,k}(\Delta\mathscr{S})(\mathbf{L}_{1,k-1}(\Delta\mathscr{S}))^{\dagger}\widetilde{P}_{k-1}^{\diamond}(z), \end{aligned} \tag{10.4}$$

$$\begin{aligned} \widetilde{Q}_{k+1}^{\diamond}(z) &:= (zI_p - \mathbf{L}_{2,k}(\Delta\mathscr{S})(\mathbf{L}_{1,k}(\Delta\mathscr{S}))^{\dagger})\widetilde{Q}_k^{\diamond}(z) \\ &\qquad - \mathbf{L}_{1,k}(\mathscr{S})(\mathbf{L}_{1,k-1}(\Delta\mathscr{S}))^{\dagger}\widetilde{Q}_{k-1}^{\diamond}(z), \end{aligned} \tag{10.5}$$

for $k \in \mathbb{Z}_{1,m-1}$, with the initial condition:

$$\widetilde{P}_0^{\diamond}(z) := I_p,\ \widetilde{P}_1^{\diamond}(z) := zI_p - s_2 s_1^{\dagger},\ \widetilde{Q}_0^{\diamond}(z) := 0_p,\ \widetilde{Q}_1^{\diamond}(z) := s_1. \tag{10.6}$$

Proposition 10.1. *Let $n \in \mathbb{Z}_{2,\infty}$ and let $\mathscr{S} \in \mathbb{C}_{\infty,n-1,H}^{p\times p}$ such that $(s_j)_{j=0}^{n-1} \in \mathcal{K}_{p,n-1}^{\geq,e}$.*

(i) *Suppose that $n = 2m$. Further suppose that $(P_k^{\diamond})_{k=0}^{m}$ and $(Q_k^{\diamond})_{k=0}^{m}$ are defined as in (10.1)–(10.3). Then $(P_k^{\diamond})_{k=0}^{m}$ is a MLOSMP of order m with respect to $\mathscr{S}$ and for each $k \in \mathbb{Z}_{0,m}$, $Q_k^{\diamond}$ is left $\mathscr{S}$-associative with $P_k^{\diamond}$.*

(ii) *Suppose that $n = 2m-1$. Further suppose that $(\widetilde{P}_k^{\diamond})_{k=0}^{m-1}$ and $(\widetilde{Q}_k^{\diamond})_{k=0}^{m-1}$ are defined as in (10.4)–(10.6). Then $(\widetilde{P}_j^{\diamond})_{j=0}^{m-1}$ is a MLOSMP of order $(m-1)$ with respect to $\Delta\mathscr{S}$ and for each $k \in \mathbb{Z}_{0,m-1}$, $\widetilde{Q}_k^{\diamond}$ is left $\Delta\mathscr{S}$-associative with $\widetilde{P}_k^{\diamond}$.*

Proof. The implication of (i) is due to Proposition 3.7, Remark 3.8 of [28] and Remark 4.27. The proof of (ii) is analogous to that of (i) and thus omitted. □

Proposition 10.2. *Let $n \in \mathbb{Z}_{2,\infty}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$ such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{\geq,e}_{p,n-1}$.*

(i) *Suppose that $n = 2m$. Further suppose that $(P^{\diamond}_k)^m_{k=0}$ and $(Q^{\diamond}_k)^m_{k=0}$ are defined as in (10.1)–(10.3). Then for $k \in \mathbb{Z}_{0,m-1}$,*

$$Q^{\diamond}_{k+1}(z)(P^{\diamond}_k)^{\vee}(z) - P^{\diamond}_{k+1}(z)(Q^{\diamond}_k)^{\vee}(z) = \mathbf{L}_{1,k}(\mathscr{S}). \tag{10.7}$$

(ii) *Suppose that $n = 2m - 1$. Further suppose that $(\widetilde{P}^{\diamond}_k)^{m-1}_{k=0}$ and $(\widetilde{Q}^{\diamond}_k)^{m-1}_{k=0}$ are defined as in (10.4)–(10.6). Then for $k \in \mathbb{Z}_{0,m-2}$,*

$$\widetilde{Q}^{\diamond}_{k+1}(z)(\widetilde{P}^{\diamond}_k)^{\vee}(z) - \widetilde{P}^{\diamond}_{k+1}(z)(\widetilde{Q}^{\diamond}_k)^{\vee}(z) = \mathbf{L}_{1,k}(\Delta\mathscr{S}). \tag{10.8}$$

Proof. The proof for (i) is by induction on k. By (10.3) it is easy to see the validity of (10.7) for the case that $k = 0$. Let $j \in \mathbb{Z}_{0,n-1}$. Suppose that (10.7) holds for the case that $k = j$.

According to Theorem 4.28 and Proposition 4.15, $(Q^{\diamond}_j)^{-1}P^{\diamond}_j$ is a Hermitian transfer function matrix, which implies

$$Q^{\diamond}_j(z)(P^{\diamond}_j)^{\vee}(z) - P^{\diamond}_j(z)(Q^{\diamond}_j)^{\vee}(z) = 0_p.$$

Then

$$\begin{aligned}
&Q^{\diamond}_{j+2}(z)(P^{\diamond}_{j+1})^{\vee}(z) - P^{\diamond}_{j+2}(z)(Q^{\diamond}_{j+1})^{\vee}(z)\\
=&\left((zI_p - \mathbf{L}_{2,j+1}(\mathscr{S})(\mathbf{L}_{1,j+1}(\mathscr{S}))^{\dagger})Q^{\diamond}_{j+1}(z) - \mathbf{L}_{1,j+1}(\mathscr{S})(\mathbf{L}_{1,j}(\mathscr{S}))^{\dagger}Q^{\diamond}_j(z)\right)(P^{\diamond}_{j+1})^{\vee}(z)-\\
&\left((zI_p - \mathbf{L}_{2,j+1}(\mathscr{S})(\mathbf{L}_{1,j+1}(\mathscr{S}))^{\dagger})P^{\diamond}_{j+1}(z) - \mathbf{L}_{1,j+1}(\mathscr{S})(\mathbf{L}_{1,j}(\mathscr{S}))^{\dagger}P^{\diamond}_j(z)\right)(Q^{\diamond}_{j+1})^{\vee}(z)\\
=&(zI_p - \mathbf{L}_{2,j+1}(\mathscr{S})(\mathbf{L}_{1,j+1}(\mathscr{S}))^{\dagger})(Q^{\diamond}_{j+1}(z)(P^{\diamond}_{j+1})^{\vee}(z) - P^{\diamond}_{j+1}(z)(Q^{\diamond}_{j+1})^{\vee}(z))+\\
&\mathbf{L}_{1,j+1}(\mathscr{S})(\mathbf{L}_{1,j}(\mathscr{S}))^{\dagger}\left(P^{\diamond}_j(z)(Q^{\diamond}_{j+1})^{\vee}(z) - Q^{\diamond}_j(z)(P^{\diamond}_{j+1})^{\vee}(z)\right)\\
=&\mathbf{L}_{1,j+1}(\mathscr{S})(\mathbf{L}_{1,j}(\mathscr{S}))^{\dagger}\mathbf{L}_{1,j}(\mathscr{S}) = \mathbf{L}_{1,j+1}(\mathscr{S}).
\end{aligned}$$

Hence, (10.7) is fulfilled.

The proof of (ii) is analogous to that of (i) and thus omitted. □

Lemma 10.3. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$ such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{\geq,e}_{p,n-1}$. Let $\mu \in \mathbb{C}^{np\times p}$ and $\lambda \in \mathbb{C}$.*

(i) *Suppose that $n = 2m$. Further suppose that $(P^{\diamond}_k)^m_{k=0}$ and $(Q^{\diamond}_k)^m_{k=0}$ are defined as in (10.1)–(10.3). Then the simultaneous equalities*

$$\mu^* P^{\diamond}_k(\lambda) = 0,\ \mu^* Q^{\diamond}_k(\lambda) = 0$$

hold for some $k \in \mathbb{Z}_{0,m}$ if and only if $\mu = 0_{p\times 1}$ or $\lambda = 0$.

(ii) *Suppose that* $n = 2m-1$. *Further suppose that* $(\widetilde{P}_k^{\diamond})_{k=0}^{m-1}$ *and* $(\widetilde{Q}_k^{\diamond})_{k=0}^{m-1}$ *are defined as in* (10.4)–(10.6). *Then the simultaneous equalities*

$$\mu^* \widetilde{P}_k^{\diamond}(\lambda) = 0, \ \mu^* \widetilde{Q}_k^{\diamond}(\lambda) = 0$$

hold for some $k \in \mathbb{Z}_{0,m-1}$ *if and only if* $\mu = 0_{p\times 1}$ *or* $\lambda = 0$.

Proof. The proof of (i): The “if” implication is trivial.

The “only if” implication: The proof is by induction on k. Suppose that there exists a $\lambda_0 \neq 0 \in \mathbb{C}$ and a nonzero vector $\mu_0 \in \mathbb{C}^{p\times 1}$ such that $\mu_0^* P_1^{\diamond}(\lambda_0) = 0$ and $\mu_0^* Q_1^{\diamond}(\lambda_0) = 0$, or equivalently,

$$s_0\mu_0 = 0, \ \lambda_0\mu_0 - s_0^{\dagger} s_1 \mu_0 = 0_{p\times 1}.$$

Using the fact that Range(s_0) ⊆Range(s_1) we have

$$\lambda_0\mu_0 = \lambda_0\mu_0 - s_0^{\dagger} s_1 \mu_0 = 0_{p\times 1}.$$

Let $j \in \mathbb{Z}_{1,n-1}$. Suppose the “only if” implication holds for $k = j$ and the simultaneous equalities

$$\mu^* P_{j+1}^{\diamond}(\lambda) = 0, \ \mu^* Q_{j+1}^{\diamond}(\lambda) = 0$$

hold. Then, by applying Proposition 10.2 we have

$$\mu^* \mathbf{L}_{1,j}(\mathscr{S}) = \mu^* \left(Q_{j+1}^{\diamond}(\lambda) P_j^{\diamond}(\bar{\lambda})^* - P_{j+1}^{\diamond}(\lambda) Q_j^{\diamond}(\bar{\lambda})^* \right) = 0.$$

Due to the fact that Range($\mathbf{L}_{2,j}(\mathscr{S})$) ⊆Range($\mathbf{L}_{1,j}(\mathscr{S})$) by [13, Lemma 2.7], subsequently we have $\mu^* \mathbf{L}_{2,j}(\mathscr{S}) = 0_p$. It follows that

$$\begin{aligned}
&\lambda\mu^* P_j^{\diamond}(\lambda) \\
=&(\lambda\mu^* - \mu^* \mathbf{L}_{2,j}(\mathscr{S})(\mathbf{L}_{1,j}(\mathscr{S}))^{\dagger}) P_j^{\diamond}(\lambda) - \mu^* \mathbf{L}_{1,j}(\mathscr{S})(\mathbf{L}_{1,j-1}(\mathscr{S}))^{\dagger} P_{j-1}^{\diamond}(\lambda) \\
=&\mu^* P_{j+1}^{\diamond}(\lambda) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\lambda\mu^* Q_j^{\diamond}(\lambda) \\
=&(\lambda\mu^* - \mu^* \mathbf{L}_{2,j}(\mathscr{S})(\mathbf{L}_{1,j}(\mathscr{S}))^{\dagger}) Q_j^{\diamond}(\lambda) - \mu^* \mathbf{L}_{1,j}(\mathscr{S})(\mathbf{L}_{1,j-1}(\mathscr{S}))^{\dagger} Q_{j-1}^{\diamond}(\lambda) \\
=&\mu^* Q_{j+1}^{\diamond}(\lambda) = 0.
\end{aligned}$$

Then, by induction we have that $\lambda = 0$ or $\mu = 0_{p\times 1}$.

The proof of (ii) is analogous to that of (i) and thus omitted.

□

Now we give an expression of particular monic quasi-stable matrix polynomials F with a given Stieltjes nonnegative definite extendable sequence being the truncated SLMP of F.

Theorem 10.4. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$ such that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{\geq,e}_{p,n-1}$.*

(i) *Suppose that $n = 2m$. Let $(P^{\diamond}_k)_{k=0}^m$ and $(Q^{\diamond}_k)_{k=0}^m$ be as in* (10.1)–(10.3) *and let, subsequently,*

$$F(z) := (-1)^m P^{\diamond}_m(-z^2) + (-1)^{m-1} z Q^{\diamond}_m(-z^2) \in \mathcal{P}_{p\times p,n,\mathbb{C}}.$$

Then F is a monic quasi-stable matrix polynomial with the $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1\rangle}$.

(ii) *Suppose that $n = 2m-1$. Let $(\widetilde{P}^{\diamond}_k)_{k=0}^{m-1}$ and $(\widetilde{Q}^{\diamond}_k)_{k=0}^{m-1}$ be as in* (10.4)–(10.6) *and let, subsequently,*

$$F(z) := (-1)^{m-1}\widetilde{Q}^{\diamond}_{m-1}(-z^2) + (-1)^{m-1}\widetilde{P}^{\diamond}_{m-1}(-z^2)(zI_p + \mathscr{S}_{\langle 0\rangle}) \in \mathcal{P}_{p\times p,n,\mathbb{C}}.$$

Then F is a monic quasi-stable matrix polynomial with the $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1\rangle}$.

Proof. The proof for "(i)": From the definition of F in (i) we see that

$$F_{\langle e\rangle}(z) = (-1)^m P^{\diamond}_m(-z),$$
$$F_{\langle o\rangle}(z) = (-1)^{m-1} Q^{\diamond}_m(-z).$$

By Propositions 7.7 and 10.1 we have that $\mathscr{S}_{\langle 2m-1\rangle}$ is the $(2m-1)$-th SLMP of F. Owing to the fact that $\mathscr{S}_{\langle 2m-1\rangle} \in \mathcal{K}^{\geq,e}_{p,2m-1}$, we have $\nu(\mathbf{H}_{m-1}) = \nu(\mathbf{H}^{\langle 1\rangle}_{m-1}) = 0$. Let $R^{\diamond}_m(z)$ be a g.l.c.d of $F_{\langle e\rangle}(-z^2)$ and $zF_{\langle o\rangle}(-z^2)$. Next we will show that $\sigma(R^{\diamond}_m) \subseteq \{0\}$. In fact, suppose that $\sigma(R^{\diamond}_m) \nsubseteq \{0\}$ and

$$R^{\diamond}_m(z)S^{\diamond}_m(z) = F_{\langle e\rangle}(-z^2), \tag{10.9}$$
$$R^{\diamond}_m(z)T^{\diamond}_m(z) = zF_{\langle o\rangle}(-z^2), \tag{10.10}$$

where $S^{\diamond}_m(z)$, $T^{\diamond}_m(z) \in \mathcal{P}_{p\times p,m,\mathbb{C}}$. Then there exists a $\lambda_0 \neq 0 \in \mathbb{C}$ and a nonzero vector $\mu_0 \in \mathbb{C}^{n\times 1}$ such that $\mu_0^* R^{\diamond}_m(\lambda_0) = 0$. Subsequently, by (10.9)–(10.10) and the fact that $\lambda_0 \neq 0$, we have $\mu_0^* P^{\diamond}_m(\lambda_0^2) = 0$ and $\mu_0^* Q^{\diamond}_m(\lambda_0^2) = 0$, which contradicts to Lemma 10.3. Hence, by Theorem 7.16,

$$\gamma'_+(F) = \nu(\mathbf{H}_{m-1}(\mathscr{S})) + \nu(\mathbf{H}^{\langle 1\rangle}_{m-1}(\mathscr{S})) + \gamma'_+(R^{\diamond}_m) = 0.$$

Therefore F is a quasi-stable matrix polynomial.

The proof for "(ii)": From the definition of F in (ii) we see that

$$F_{\langle e\rangle}(z) = (-1)^{m-1}\widetilde{Q}^{\diamond}_{m-1}(-z) + (-1)^{m-1}\widetilde{P}^{\diamond}_{m-1}(-z)\mathscr{S}_{\langle 0\rangle},$$
$$F_{\langle o\rangle}(z) = (-1)^{m-1}\widetilde{P}^{\diamond}_{m-1}(-z).$$

By Propositions 7.8 and 10.1 we have that $\mathscr{S}_{\langle 2m-2\rangle}$ is the $(2m-2)$-th SLMP of F. Since $\mathscr{S}_{\langle 2m-2\rangle} \in \mathcal{K}^{\geq,e}_{p,2m-2}$, we have $\nu(\mathbf{H}_{m-1}(\mathscr{S})) = \nu(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) = 0$. Let $\widetilde{R}^{\diamond}_{m-1}(z)$

be a g.l.c.d of $F_{\langle e\rangle}(-z^2)$ and $zF_{\langle o\rangle}(-z^2)$. Next, we will show that $\sigma(\widetilde{R}^{\diamond}_{m-1}) \subseteq \{0\}$. In fact, if $\sigma(\widetilde{R}^{\diamond}_{m-1}) \nsubseteq \{0\}$ and

$$\widetilde{R}^{\diamond}_{m-1}(z)\widetilde{S}^{\diamond}_{m-1}(z) = F_{\langle e\rangle}(-z^2), \tag{10.11}$$

$$\widetilde{R}^{\diamond}_{m-1}(z)\widetilde{T}^{\diamond}_{m-1}(z) = zF_{\langle o\rangle}(-z^2), \tag{10.12}$$

where $\widetilde{S}^{\diamond}_{m-1}(z)$, $\widetilde{T}^{\diamond}_{m-1}(z) \in \mathcal{P}_{p\times p,m,\mathbb{C}}$, then there exists a $\lambda_0 \neq 0 \in \mathbb{C}$ and a nonzero vector $\mu_0 \in \mathbb{C}^{n\times 1}$ such that $\mu_0^*\widetilde{R}^{\diamond}_{m-1}(\lambda_0) = 0$. Subsequently, by (10.11)–(10.12) and the fact that $\lambda_0 \neq 0$, we have

$$\mu_0^*\widetilde{P}^{\diamond}_{m-1}(\lambda_0^2) = (-1)^{m-1}\lambda_0^{-1}\mu_0^*\widetilde{R}^{\diamond}_{m-1}(\lambda_0^2)S^{\diamond}_{m-1}(\lambda_0^2) = 0,$$

and

$$\begin{aligned}
&\mu_0^*\widetilde{Q}^{\diamond}_{m-1}(\lambda_0^2)\\
=&\mu_0^*\left(\widetilde{Q}^{\diamond}_{m-1}(\lambda_0^2) + \widetilde{P}^{\diamond}_{m-1}(\lambda_0^2)\mathscr{S}_{\langle 0\rangle}\right) - \mu_0^*\widetilde{P}^{\diamond}_{m-1}(\lambda_0^2)\mathscr{S}_{\langle 0\rangle}\\
=&(-1)^{m-1}\mu_0^*\left(\widetilde{R}^{\diamond}_{m-1}(\lambda_0^2)\widetilde{S}^{\diamond}_{m-1}(\lambda_0^2)\right) - (-1)^{m-1}\lambda_0^{-1}\mu_0^*\widetilde{R}^{\diamond}_{m-1}(\lambda_0^2)S^{\diamond}_{m-1}(\lambda_0^2)\mathscr{S}_{\langle 0\rangle}\\
=&0,
\end{aligned}$$

which contradicts to Lemma 10.3. Hence, by Theorem 7.16,

$$\gamma'_+(F) = \nu(\mathbf{H}_{m-1}(\mathscr{S})) + \nu(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) + \gamma'_+(\widetilde{R}^{\diamond}_{m-1}) = 0.$$

Therefore, F is a quasi-stable matrix polynomial.

□

In what follows we give a correspondence between the existence of a monic quasi-stable matrix polynomial and Stieltjes nonnegative extendable sequences.

Theorem 10.5. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$. Then there exists a monic quasi-stable matrix polynomial $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ with the $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1\rangle}$ if and only if $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{\geq,e}_{p,n-1}$.*

Proof. The "if" implication is an immediate consequence of Theorem 10.4.

The "only if" implication: Case I: $n = 2m$. Suppose that $F(z) \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ is a monic quasi-stable matrix polynomial with the SLMP $(s_j)_{j=0}^{n-1}$. Then by applying Theorem 7.19 we have that

$$0 \leq \nu(\mathbf{H}_{m-1}(\mathscr{S})) \leq \gamma'_+(F) = 0,\ 0 \leq \nu(\mathbf{H}^{\langle 1\rangle}_{m-1}(\mathscr{S})) \leq \gamma'_+(F) = 0.$$

Combinating this with (iv) of Proposition 7.5 and [28, Lemma 2.16], it follows that $(s_j)_{j=0}^{2m-1} \in \mathcal{H}^{\geq,e}_{p,2m-1}$ and $(s_{j+1})_{j=0}^{2m-2} \in \mathcal{H}^{\geq}_{p,2m-2}$, or equivalently, $(s_j)_{j=0}^{2m-1} \in \mathcal{K}^{\geq,e}_{p,2m-1}$.

Case II: $n = 2m-1$. Suppose that $F(z) \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ is a monic quasi-stable matrix polynomial with the SLMP $(s_j)_{j=0}^{n-1}$. Then by applying Theorem 7.19 we have that

$$0 \leq \nu(\mathbf{H}_{m-1}(\mathscr{S})) \leq \gamma'_+(F) = 0,\ 0 \leq \nu(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) \leq \gamma'_+(F) = 0.$$

Accordingly, by a combination of (v) of Proposition 7.5 and [28, Lemma 2.16], one can see that $(s_j)_{j=0}^{2m-2} \in \mathcal{H}_{p,2m-2}^{\geq}$ and $(s_{j+1})_{j=0}^{2m-3} \in \mathcal{H}_{p,2m-3}^{\geq,e}$, or equivalently, $(s_j)_{j=0}^{2m-2} \in \mathcal{K}_{p,2m-2}^{\geq,e}$. □

The equivalence between the existence of a monic quasi-stable matrix polynomial and the solvability of truncated Stieltjes moment problem is also established.

Theorem 10.6. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}_{\infty,n-1,H}^{p\times p}$. Then the following statements are equivalent:*

(i) $\mathcal{M}_{\geq}^{p}[[0,\infty);\mathscr{S}_{\langle n-1\rangle},=] \neq \emptyset$.

(ii) *There exists a monic quasi-stable matrix polynomial $F_1 \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ with the $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1\rangle}$.*

(iii) *There exists a monic quasi-stable matrix polynomial $F_2 \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ with the $(n-1)$-th SRMP $\mathscr{S}_{\langle n-1\rangle}$.*

(iv) $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}_{p,n-1}^{\geq,e}$.

Proof. The equivalences "(i)$\Longleftrightarrow$ (iv)", "(ii) $\Longleftrightarrow$ (iii)" and "(ii) $\Longleftrightarrow$ (iv)" follow from Theorem 3.8, Remark 7.3 and Theorem 10.5, respectively. □

10.2 Particular monic quasi-stable matrix polynomials and multiple Nevanlinna-Pick interpolation in the Stieltjes class

This section is a continuation of Section 10.1. We extend our investigation of the above-studied particular quasi-stable matrix polynomials to the connections between such matrix polynomials and the Nevanlinna-Pick interpolation problem in the Stieltjes class. This interpolation problem was intensively studied in the literature (see Kreĭn/Nudelman [58] for the scalar case and Bolotnikov and Sakhnovich [9], Dyukarev [25, 26], and Dyukarev/Katsnelson [29] and Chen/Li [16] for the matrix case).

Our motivation comes from a series of researches conducted by the research group around Gongning Chen and Yongjian Hu from Beijing Normal University, concerning the inner connection between various moment problems and Nevanlinna-Pick interpolation problems in the scalar case and the matrix case (see e.g. [13]–[16], [45], [46]). The most important basic tool in their works is the approach of (block) Hankel vectors.

Making use of an elegant correspondence between Stieltjes moment problems and Nevanlinna-Pick interpolation problems for matrix-valued Stieltjes functions by Chen/Li [16], we establish an equivalent condition between the existence of monic quasi-stable matrix polynomials with a given SLMP or SRMP and the solvability of the multiple Nevanlinna-Pick interpolation in the Stieltjes class.

Recall that a $\mathbb{C}^{p\times p}$-valued function F analytic on the open upper half-plane $\mathbb{C}_U$ is said to be a *Nevanlinna function* if $\mathrm{Im}F(z) \geq 0_p$ for z for $z \in \mathbb{C}_U$. In addition, F is a *Stieltjes function* if F is a Nevanlinna function and F is analytic on $\mathbb{C} \setminus [0, +\infty)$ and, for each $z \in (-\infty, 0)$, $F(z) \geq 0_p$. We denote by $\mathfrak{S}_p$ the set of all $p \times p$-valued Stieltjes functions. Each $F \in \mathfrak{S}_p$ permits an integral representation (see [58])

$$F(z) = \alpha + \int_{[0,+\infty)} \frac{\theta(\mathrm{d}u)}{u - z}, \tag{10.13}$$

where $\alpha \in \mathbb{C}_H^{p\times p}$ and $\theta \in \mathcal{M}_{\geq}^p([0, +\infty))$ such that

$$\mathrm{trace} \int_{[0,+\infty)} (1 + u)^{-1}\theta(\mathrm{d}u) < +\infty.$$

In particular, we denote by $\mathfrak{S}_p^0$ the subset of $\mathfrak{S}_p$ that consists of all $F \in \mathfrak{S}_p$ admitting the representation (10.13) with $\alpha = 0_p$. Now we introduce the multiple Nevanlinna-Pick interpolation problem in the class $\mathfrak{S}_p^0$:

Problem NP($\mathfrak{S}_p^0$)$[(\lambda_j)_{j=1}^s, (\rho_j)_{j=1}^s, (D_{jk})_{j=1,k=0}^{s,\rho_j-1}]$ Let $d \in \mathbb{Z}_{1,\infty}$ and let $s \in \mathbb{Z}_{d,\infty}$. Let $\lambda_j \in \mathbb{C}_U$ for each $j \in \mathbb{Z}_{1,d}$ and $\lambda_j \in (-\infty, 0)$ for each $j \in \mathbb{Z}_{d+1,s}$, which are distinct, with multiplicities $\rho_1, \cdots, \rho_s$, respectively. Moreover, let, for each $j \in \mathbb{Z}_{1,d}$ and $k \in \mathbb{Z}_{0,\rho_j-1}$, $D_{jk} \in \mathbb{C}^{p\times p}$, and let, for each $j \in \mathbb{Z}_{d+1,s}$ and $k \in \mathbb{Z}_{0,\rho_j-1}$, $D_{jk} \in \mathbb{C}_H^{p\times p}$. Describe the set $\mathcal{NP}(\mathfrak{S}_p^0)[(\lambda_j)_{j=1}^s, (\rho_j)_{j=1}^s, (D_{jk})_{j=1,k=0}^{s,\rho_j-1}]$ of all $F \in \mathfrak{S}_p^0$ such that

$$\frac{1}{k!}F^{(k)}(\lambda_j) = D_{jk},\ j \in \mathbb{Z}_{1,s},\ k \in \mathbb{Z}_{0,\rho_j-1}.$$

Let

$$\widetilde{N} := 2\left(\sum_{j=1}^{d} \rho_j\right) + \sum_{j=d+1}^{s} \rho_j \tag{10.14}$$

is the number of all interpolation nodes with multiplicities. Then there exists a unique matrix polynomial $\Omega \in \mathcal{P}_{p\times p,\mathbb{C}}$ of degree at most $\widetilde{N}-1$, which is referred to the *Hermitian interpolation matrix polynomial of* $\mathrm{NP}(\mathfrak{S}_p^0)[(\lambda_j)_{j=1}^s, (\rho_j)_{j=1}^s, (D_{jk})_{j=1,k=0}^{s,\rho_j-1}]$ satisfying the following conditions

$$\frac{1}{k!}\Omega^{(k)}(\lambda_j) = D_{jk},\ \frac{1}{k!}\Omega^{(k)}(\overline{\lambda}_j) = D_{jk},\ j \in \mathbb{Z}_{1,\theta},\ k \in \mathbb{Z}_{0,\rho_j-1}.$$

Now put

$$a(z) := \prod_{j=1}^{d}(z - \lambda_j)^{\rho_j}(z - \overline{\lambda}_j)^{\rho_j} \prod_{j=d+1}^{s} (z - \lambda_j)^{\rho_j}.$$

Define in turn,

$$G(z) = \frac{\Omega(z)}{a(z)}, \quad z \in \mathbb{C}.$$

Suppose that G is represented by the following Laurent series at infinity,

$$G(z) = \sum_{l=0}^{\infty} z^{-(l+1)} s_l, \quad z \in \mathbb{C}.$$

Then $s_j \in \mathbb{C}_H^{p\times p}$ for each $j \in \mathbb{Z}_{0,\widetilde{N}-1}$. We call $(s_j)_{j=0}^{\widetilde{N}-1}$ the *block Hankel vector of Problem* $\mathrm{NP}[(\lambda_j)_{j=1}^s, (\rho_j)_{j=1}^s, (D_{jk})_{j=1,k=0}^{s,\rho_j-1}]$. An immediate consequence of [16, Theorem 2.2] is the one-to-one correspondence between the multiple Nevanlinna-Pick interpolation problem in the class $\mathfrak{S}_p^0$ and the related Stieltjes matrix moment problem.

Lemma 10.7. *Let $d \in \mathbb{Z}_{1,\infty}$ and let $s \in \mathbb{Z}_{d,\infty}$. Let $\lambda_j \in \mathbb{C}_U$ for each $j \in \mathbb{Z}_{1,d}$ and $\lambda_j \in (-\infty, 0)$ for each $j \in \mathbb{Z}_{d+1,s}$, which are distinct, with multiplicities $\rho_1, \cdots, \rho_s$, respectively. Moreover, let, for each $j \in \mathbb{Z}_{1,d}$ and $k \in \mathbb{Z}_{0,\rho_j-1}$, $D_{jk} \in \mathbb{C}^{p\times p}$, and let, for each $j \in \mathbb{Z}_{d+1,s}$ and $k \in \mathbb{Z}_{0,\rho_j-1}$, $D_{jk} \in \mathbb{C}_H^{p\times p}$. Let $\widetilde{N}$ be as in* (10.14) *and let $\mathscr{S}_{\widetilde{N}-1}$ the block-Hankel vector of Problem $NP[(\lambda_j)_{j=1}^s, (\rho_j)_{j=1}^s, (D_{jk})_{j=1,k=0}^{s,\rho_j-1}]$. Then $\mathcal{NP}(\mathfrak{S}_p^0)[(\lambda_j)_{j=1}^s, (\rho_j)_{j=1}^s, (D_{jk})_{j=1,k=0}^{s,\rho_j-1}] \neq \emptyset$ if and only if $\mathcal{M}_{\geq}^p[[0,\infty); \mathscr{S}_{\widetilde{N}-1}, =] \neq \emptyset$. More precisely, suppose that $F \in \mathcal{NP}(\mathfrak{S}_p^0)[(\lambda_j)_{j=1}^s, (\rho_j)_{j=1}^s, (D_{jk})_{j=1,k=0}^{s,\rho_j-1}]$ and F admits an integral representation of the form* (10.13)*. Further suppose that*

$$\tau(u) := \int_{[0,u]} A(t)^{-1} \theta(\mathrm{d}u),\ u \in [0, +\infty).$$

Then $\tau \in \mathcal{M}_{\geq}^p[[0,\infty); \mathscr{S}_{\widetilde{N}-1}, =]$. Conversely, suppose that $\tau \in \mathcal{M}_{\geq}^p[[0,\infty); \mathscr{S}_{\widetilde{N}-1}, =]$. Further suppose that $F \in \mathfrak{S}_p^0$ and F admits an integral representation of the form (10.13)*, where*

$$\theta(u) := \int_{[0,u]} A(t) \tau(\mathrm{d}u),\ u \in [0, +\infty).$$

Then $F \in \mathcal{NP}(\mathfrak{S}_p^0)[(\lambda_j)_{j=1}^s, (\rho_j)_{j=1}^s, (D_{jk})_{j=1,k=0}^{s,\rho_j-1}]$.

In view of Theorem 10.6 and Lemma 10.7, we establish the connections between the solvability of the multiple Nevanlinna-Pick interpolation problem in the class $\mathfrak{S}_p^0$ and the existence of a certain quasi-stable matrix polynomial.

Theorem 10.8. *Let d, s, $(\lambda_j)_{j=1}^s$, $(\rho_j)_{j=1}^s$, $(D_{jk})_{j=1,k=0}^{s,\rho_j-1}$, $\widetilde{N}$ and $\mathscr{S}_{\widetilde{N}-1}$ be as in Lemma 10.7. Then*

(i) $\mathcal{NP}(\mathfrak{S}_p^0)[(\lambda_j)_{j=1}^s, (\rho_j)_{j=1}^s, (D_{jk})_{j=1,k=0}^{s,\rho_j-1}] \neq \emptyset$.

(ii) *There exists a monic quasi-stable matrix polynomial $F_1 \in \mathcal{P}_{p\times p,\widetilde{N},\mathbb{C}}$ with the $(\widetilde{N}-1)$-th SLMP $\mathscr{S}_{\widetilde{N}-1}$.*

(iii) *There exists a monic quasi-stable matrix polynomial $F_2 \in \mathcal{P}_{p\times p,\widetilde{N},\mathbb{C}}$ with the $(\widetilde{N}-1)$-th SRMP $\mathscr{S}_{\widetilde{N}-1}$.*

(iv) $\mathscr{S}_{\widetilde{N}-1} \in \mathcal{K}_{p,\widetilde{N}-1}^{\geq,e}$.

10.3 General description of monic quasi-stable matrix polynomials

In this section, we characterize some essential features for the whole set of quasi-stable matrix polynomials, which as a special case also includes the particular monic quasi-stable matrix polynomials considered in Sections 10.1 and 10.2.

In the previous section, we obtained a correspondence between the existence of a monic quasi-stable matrix polynomial and Stieltjes nonnegative definite extendable sequences. However, given a monic matrix polynomial F of degree n, the requirement that the $(n-1)$-th SRMP or SLMP of F is a Stieltjes nonnegative definite extendable sequence is not sufficient to identify F to be a quasi-stable matrix polynomial. We proceed by looking for extra conditions imposed on F so that F is a quasi-stable matrix polynomial.

Theorem 10.9. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic with the $(n-1)$-th SRMP $\mathscr{S}_{\langle n-1\rangle}$. Then F is a quasi-stable matrix polynomial if and only if $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{\geq,e}_{p,n-1}$ and $\sigma(\widetilde{F}) \subseteq (-\infty,0]$, where $\widetilde{F}$ is a g.r.c.d of $F_{\langle e\rangle}$ and $F_{\langle o\rangle}$.*

Proof. The "only if" implication: suppose that F is a quasi-stable matrix polynomial. The "only if" implication of Theorem 10.5 shows that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{\geq,e}_{p,n-1}$. On account of Theorem 7.16 and the fact that $\gamma'_+(F) = 0$, we have $\gamma_-(F^\diamond) = 0$. By (7.15) and (7.16), it follows that

$$\gamma_+(\widehat{F}) = \gamma_-(\widehat{F}) = \gamma_-(F^\diamond) = 0. \tag{10.15}$$

According to Theorem 7.17 and its proof, we deduce for the case that $n = 2m$ with $m \in \mathbb{N}$,

$$\deg(\det\widetilde{F}(z)) = \delta(\mathbf{H}_{m-1}(\mathscr{S})) = \gamma_{(-\infty,0]}(\widetilde{F}),$$

where the first equality comes from (7.19) and the second equality comes from (10.15) and (7.20), and for the case that $n = 2m-1$ with $m \in \mathbb{N}$,

$$\deg(\det\widetilde{F}(z)) = \delta(\mathbf{H}^{\langle 1\rangle}_{m-2}(\mathscr{S})) = \gamma_{(-\infty,0]}(\widetilde{F}),$$

where the first equality comes from (7.22) and the second equality comes from (10.15) and (7.23). Hence $\sigma(\widetilde{F}) \in (-\infty,0]$.

The "if" implication: suppose that $\mathscr{S}_{\langle n-1\rangle} \in \mathcal{K}^{\geq,e}_{p,n-1}$ and $\sigma(\widetilde{F}) \in (-\infty,0]$. Then

$$\nu(\mathbf{H}_{[\frac{n-1}{2}]}(\mathscr{S})) = 0, \quad \nu(\mathbf{H}^{\langle 1\rangle}_{[\frac{n}{2}]-1}(\mathscr{S})) = 0 \tag{10.16}$$

and

$$\gamma_{(-\infty,0]}(\widetilde{F}) = \deg(\det\widetilde{F}(z)).$$

By using Theorem 7.17 and its proof (particularly (7.19) and (7.22)) again, we have for the case that $n = 2m$ with $m \in \mathbb{N}$,

$$\delta(\mathbf{H}_{m-1}(\mathscr{S})) = \deg(\det\widetilde{F}(z)) = \gamma_{(-\infty,0]}(\widetilde{F}) \tag{10.17}$$

and for the case that $n = 2m - 1$ with $m \in \mathbb{N}$,

$$\delta(\mathbf{H}^{\langle 1 \rangle}_{m-2}(\mathscr{S})) = \deg(\det \widetilde{F}(z)) = \gamma_{(-\infty,0]}(\widetilde{F}). \tag{10.18}$$

Then a combination of (10.16)–(10.18) and Theorem 7.17 gives that $\gamma'_{+}(F) = 0$, or equivalently, F is a monic quasi-stable matrix polynomial. □

By substituting $F^{\vee}$ for F in Theorem 10.9, we obtain the following dual result from Remark 7.3 and Proposition 2.18.

Theorem 10.10. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be monic with the $(n-1)$-th SLMP $\mathscr{S}_{\langle n-1 \rangle}$. Then F is a quasi-stable matrix polynomial if and only if $\mathscr{S}_{\langle n-1 \rangle} \in \mathcal{K}^{\geq,e}_{p,n-1}$ and $\sigma(\widetilde{F}) \in (-\infty, 0]$, where $\widetilde{F}$ is a g.l.c.d of $F_{\langle e \rangle}$ and $F_{\langle o \rangle}$.*

In view of Theorems 10.4 and 10.10, the following consequence reveals the zero location of a g.r.c.d of two matrix polynomials related to a particular MLOSMP.

Corollary 10.11. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$ such that $\mathscr{S}_{\langle n-1 \rangle} \in \mathcal{K}^{\geq,e}_{p,n-1}$.*

(i) *Suppose that $n = 2m$. Further suppose that $(P^{\diamond}_k)^m_{k=0}$ and $(Q^{\diamond}_k)^m_{k=0}$ are defined as in* (10.1)–(10.3)*. Then $\sigma(R^{\diamond}_m) \subseteq [0, +\infty)$, where $R^{\diamond}_m$ is a g.l.c.d of $P^{\diamond}_m$ and $Q^{\diamond}_m$.*

(ii) *Suppose that $n = 2m - 1$. Further suppose that $(\widetilde{P}^{\diamond}_k)^{m-1}_{k=0}$ and $(\widetilde{Q}^{\diamond}_k)^{m-1}_{k=0}$ are defined as in* (10.4)–(10.6)*. Then $\sigma(\widetilde{R}^{\diamond}_m) \subseteq [0, +\infty)$, where $\widetilde{R}^{\diamond}_m$ is a g.l.c.d of $\widetilde{P}^{\diamond}_m$ and $\widetilde{Q}^{\diamond}_m$.*

It should be mentioned that in the above discussion of the relation between a matrix polynomial $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ and its SRMP (resp. SLMP) $\mathscr{S}$, we concerned only the $(n-1)$-th SRMP (resp. SLMP) $\mathscr{S}_{\langle n-1 \rangle}$ containing the first $(n-1)$ elements of $\mathscr{S}$. The reason why we discard the whole SRMP (resp. SLMP) is that F is uniquely determined by $\mathscr{S}_{\langle n-1 \rangle}$ and $\mathscr{S}$ can be represented via these preceeding $(n-1)$-th elements $\mathscr{S}_{\langle n-1 \rangle}$ (See Proposition 7.5). In the following, however, we pay much attention to the SRMP (resp. SLMP) of F when F is particularly chosen to be quasi-stable and, indeed, complete the whole picture of the SRMP (resp. SLMP) of F. Before that, an introduction of a further important subclass of $\mathcal{K}^{\geq}_{p,n}$ is necessary.

Definition 10.12. Let $n \in \mathbb{N}_0$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty}$ such that $\mathscr{S}_{\langle n \rangle} \in \mathcal{K}^{\geq}_{p,n}$. Then $\mathscr{S}_{\langle n \rangle}$ is called *completely degenerate* if $\mathbf{L}_{1,m}(\mathscr{S}) = 0_p$ in the case $n = 2m$ with some $m \in \mathbb{N}_0$ or if $\mathbf{L}_{1,m}(\Delta\mathscr{S}) = 0_p$ in the case $n = 2m+1$ with some $m \in \mathbb{N}_0$. The set $\mathcal{K}^{\geq,\mathrm{cd}}_{p,n}$ of all completely degenerate sequences belonging to $\mathcal{K}^{\geq}_{p,n}$ is a subset of $\mathcal{K}^{\geq,e}_{p,n}$ (see [31, Proposition 5.9, p. 231]). Further, suppose that $\mathscr{S} \in \mathcal{K}^{\geq}_{p,\infty}$. The sequence $\mathscr{S}$ is called *completely degenerate of order n* if $\mathscr{S}_{\langle n \rangle}$ is completely degenerate. By $\mathcal{K}^{\geq,\mathrm{cd},n}_{p,\infty}$ we denote the set of all Stieltjes nonnegative definite sequences which are completely degenerate of order n.

Theorem 10.13. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be a monic quasi-stable matrix polynomial.*

(i) *Suppose that* $\mathscr{S}$ *is the SRMP of* F. *Then* $\mathscr{S} \in \mathcal{K}_{p,\infty}^{\geq,\mathrm{cd},n}$.

(ii) *Suppose that* $\mathscr{S}$ *is the SLMP of* F. *Then* $\mathscr{S} \in \mathcal{K}_{p,\infty}^{\geq,\mathrm{cd},n}$.

Proof. We give a proof for (i).

Case I: $n = 1$. Let $\mathscr{S}_{\langle 0\rangle} := s_0$. Then (i) of Proposition 7.5 shows that $\mathscr{S} = (s_0, 0_p, \ldots)$. So $\mathscr{S} \in \mathcal{K}_{p,\infty}^{\geq,\mathrm{cd},1}$.

Case II: $n = 2m$ and $m \in \mathbb{N}$. Let $k \in \mathbb{Z}_{m,\infty}$ and let $F_{\langle e,k\rangle}(z) := z^{k-m}F_{\langle e\rangle}(z)$. By applying Theorem 10.6, we can show that $\mathscr{S}_{\langle 2m-1\rangle} \in \mathcal{K}_{p,2m-1}^{\geq,e}$. Then both $\mathbf{H}_{m-1}(\mathscr{S})$ and $\mathbf{H}_{m-1}^{\langle 1\rangle}(\mathscr{S})$ are nonnegative definite. For each $k \in \mathbb{N}$, let $\mathcal{I}_k$ be as in (7.10). From (ii) of Proposition 7.5, we have for $l \in \mathbb{N}_0$,

$$\mathbf{H}_{k-1}^{\langle l+1\rangle}(\mathscr{S}) = -\mathbf{H}_{k-1}^{\langle l\rangle}(\mathscr{S})\mathcal{I}_{k-1}(\mathbf{C}_{F_{\langle e,k\rangle}}^{\langle 2\rangle})\mathcal{I}_{k-1}.$$

It follows that

$$\mathbf{H}_{k-1}^{\langle l\rangle}(\mathscr{S})\mathbf{X}_{k-1} = \mathbf{Y}_{k+l,2k+l-1}(\mathscr{S}), \tag{10.19}$$

where

$$\mathbf{X}_{k-1} := -\mathcal{I}_{k-1}(\mathbf{C}_{F_{\langle e,k\rangle}}^{\langle 2\rangle})\mathcal{I}_{k-1}\begin{pmatrix}0_p\\ \vdots\\ 0_p\\ I_p\end{pmatrix}.$$

Let $\mathscr{S} := (s_j)_{j=0}^{\infty}$. Then

$$\begin{aligned}
s_{2k+l} &= (0_p, \ldots, 0_p, I_p)\,\mathbf{Z}_{k+l+1,2k+l}(\mathscr{S})\\
&= (0_p, \ldots, 0_p, I_p)\,\mathbf{H}_{k-1}^{\langle l+1\rangle}(\mathscr{S})\mathbf{X}_{k-1}\\
&= \mathbf{Z}_{k+l,2k+l-1}(\mathscr{S})\mathbf{X}_{k-1}\\
&= \mathbf{X}_{k-1}^{*}\mathbf{H}_{k-1}^{\langle l\rangle}(\mathscr{S})\mathbf{X}_{k-1}\\
&= \mathbf{X}_{k-1}^{*}\mathbf{H}_{k-1}^{\langle l\rangle}(\mathscr{S})\left(\mathbf{H}_{k-1}^{\langle l\rangle}(\mathscr{S})\right)^{\dagger}\mathbf{H}_{k-1}^{\langle l\rangle}(\mathscr{S})\mathbf{X}_{k-1}\\
&= \mathbf{Z}_{k+l,2k+l-1}(\mathscr{S})\left(\mathbf{H}_{k-1}^{\langle l\rangle}(\mathscr{S})\right)^{\dagger}\mathbf{Y}_{k+l,2k+l-1}(\mathscr{S}),
\end{aligned}$$

where the 2nd equation, the 4th equation and the last equation are due to (10.19). Hence $\mathbf{L}_{1,k}(\mathscr{S}) = \mathbf{L}_{1,k}(\Delta\mathscr{S}) = 0_p$ and then both $\mathbf{H}_k(\mathscr{S})$ and $\mathbf{H}_k^{\langle 1\rangle}(\mathscr{S})$ are nonnegative definite. Therefore $\mathscr{S} \in \mathcal{K}_{p,\infty}^{\geq,\mathrm{cd},2m}$.

Case III: $n = 2m+1$ and $m \in \mathbb{N}$. Let $k \in \mathbb{Z}_{m,\infty}$ and let $F_{\langle o,k\rangle}(z) := z^{k-m+1}F_{\langle o\rangle}(z)$. By applying Theorem 10.6, we can show that $\mathscr{S}_{\langle 2m\rangle} \in \mathcal{K}_{p,2m}^{\geq,e}$. Then both $\mathbf{H}_m(\mathscr{S})$ and $\mathbf{H}_{m-1}^{\langle 1\rangle}(\mathscr{S})$ are nonnegative definite. For each $k \in \mathbb{N}$, let $\mathcal{I}_k$ be as in (7.10). From (iii) of Proposition 7.5, we have for $l \in \mathbb{N}_0$,

$$\mathbf{H}_{k}^{\langle l+1\rangle}(\mathscr{S}) = -\mathbf{H}_{k}^{\langle l\rangle}(\mathscr{S})\mathcal{I}_{k}(\mathbf{C}_{F_{\langle o,k\rangle}}^{\langle 2\rangle})\mathcal{I}_{k}.$$

It follows that

$$\mathbf{H}_k^{\langle l\rangle}(\mathscr{S})\widetilde{\mathbf{X}}_k = \mathbf{H}_k^{\langle l+1\rangle}(\mathscr{S})\begin{pmatrix} 0_p \\ \vdots \\ 0_p \\ I_p \end{pmatrix} = \mathbf{Y}_{k+l+1,2k+l+1}(\mathscr{S}), \tag{10.20}$$

where

$$\widetilde{\mathbf{X}}_k := -\mathcal{I}_k(\mathbf{C}^{\langle 2\rangle}_{F_{\langle o,k\rangle}})\mathcal{I}_k\begin{pmatrix} 0_p \\ \vdots \\ 0_p \\ I_p \end{pmatrix}.$$

Let $\mathscr{S} := (s_j)_{j\in\mathbb{N}_0}$. Then

$$\begin{aligned} s_{2k+l+2} &= (0_p, \dots, 0_p, I_p)\,\mathbf{Y}_{k+l+2,2k+l+2}(\mathscr{S}) \\ &= (0_p, \dots, 0_p, I_p)\,\mathbf{H}_k^{\langle l+1\rangle}(\mathscr{S})\widetilde{\mathbf{X}}_k \\ &= \mathbf{Z}_{k+l+1,2k+l+1}(\mathscr{S})\widetilde{\mathbf{X}}_k \\ &= \widetilde{\mathbf{X}}_k^*\mathbf{H}_k^{\langle l\rangle}(\mathscr{S})\widetilde{\mathbf{X}}_k \\ &= \widetilde{\mathbf{X}}_k^*\mathbf{H}_k^{\langle l\rangle}(\mathscr{S})\left(\mathbf{H}_k^{\langle l\rangle}(\mathscr{S})\right)^\dagger \mathbf{H}_k^{\langle l\rangle}(\mathscr{S})\widetilde{\mathbf{X}}_k \\ &= \mathbf{Z}_{k+l+1,2k+l+1}(\mathscr{S})\left(\mathbf{H}_k^{\langle l\rangle}(\mathscr{S})\right)^\dagger \mathbf{Y}_{k+l+1,2k+l+1}(\mathscr{S}), \end{aligned}$$

where the 2nd equation, the 4th equation and the last equation are due to (10.20). Hence $\mathbf{L}_{1,k}(\mathscr{S}) = \mathbf{L}_{1,k}(\Delta\mathscr{S}) = 0_p$ and then both $\mathbf{H}_k(\mathscr{S})$ and $\mathbf{H}_k^{\langle 1\rangle}(\mathscr{S})$ are nonnegative definite. Therefore $\mathscr{S} \in \mathcal{K}^{\geq,\mathrm{cd},2m+1}_{p,\infty}$.

The proof for (ii) is analogous and omitted. □

Corollary 10.14. *Let $n \in \mathbb{N}$ and let $\mathscr{S} \in \mathbb{C}^{p\times p}_{\infty,n-1,H}$. Let $F \in \mathcal{P}_{p\times p,n,\mathbb{C}}$ be a monic Hurwitz matrix polynomial.*

(i) *Suppose that $\mathscr{S}$ is the SRMP of F. Then $\mathscr{S} \in \mathcal{K}^{\geq,\mathrm{cd},n}_{p,\infty}$.*

(ii) *Suppose that $\mathscr{S}$ is the SLMP of F. Then $\mathscr{S} \in \mathcal{K}^{\geq,\mathrm{cd},n}_{p,\infty}$.*

List of terms

List of notations

$A^\dagger$ The Moore-Penrose inverse of A, page 7

$\frac{A}{B} := A \cdot B^{-1}$ Page 67

$B_{\widetilde{M},M}(L, \widetilde{L})$ Definition 5.1, page 33

$\mathbb{C}$ The set of all complex numbers, page 7

$\mathbf{C}_F^{\langle 1 \rangle}$ The first companion matrix of F, Definition 4.10, page 23

$\mathbf{C}_F^{\langle 2 \rangle}$ The second companion matrix of F, Definition 4.10, page 23

$\mathbb{C}^{p\times q}$ The set of all complex $p \times q$ matrices, page 7

$\mathbb{C}_H^{p\times q}$ The set of all complex Hermitian $p \times p$ matrices, page 7

$\mathbb{C}_>^{p\times q}$ The set of all positive definite Hermitian $p \times p$ matrices, page 7

$\mathbb{C}_\kappa^{p\times q}$ The set of all matrix sequences over $\mathbb{C}^{p\times q}$ indexed by $\mathbb{Z}_{0,\kappa}$, page 15

$\mathbb{C}_{\kappa,\infty}^{p\times q}$ Page 15

$\mathbb{C}_L$ The open lower half plane, page 36

$\mathbb{C}_U$ The open upper half plane, page 36

$\mathbb{C}_+$ The open right half plane, page 36

$\mathbb{C}_-$ The open left half plane, page 36

$\deg F$ The degree of F, page 7

$\deg_k F$ The k-th column degree of F, page 7

$\mathcal{D}_{\langle A_{21}, A_{22} \rangle} := \{X \in \mathbb{C}^{p\times p} : \det(A_{21}X + A_{22}) \neq 0\}$ Page 69

$\Delta\mathscr{S}$ Page 15

$F^\vee$ Page 7

$F \circ f$ The composite function of F and f, Definition 2.22, page 13

$F_{\langle e \rangle}$ The even part of F, Definition 7.1, page 49

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